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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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OF THE REGISTRAR

7 PM MAY 1930

SOME PROPERTIES OF DEVELOPABLE SURFACES.

By C. N. SRINIVASIENGAR, M.Sc.
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Central College, Bangalore.

1. Preliminary. On a surface referred to a system of parametric curves p = const. and q = const. consider a point at which the fundamental magnitudes of the second order L, M, N all vanish. At such a point, the equation

$$L\,dp^2 + 2M\,dp\,dq + N\,dq^2 = 0$$

determining the inflexional directions becomes an identity. There are two possibilities. *Either the point is a double point,* or else the indicatrix at the point is a plane cubic instead of a conic.* In either case, there exist a singly infinite number of straight lines which have contact of the second order with the surface at (p, q).

It may be proper here to describe the nature of points at which the indicatrices are cubics, as this type of singularity does not appear to have received any attention hitherto.

When the origin is referred to such a point O, the Cartesian equation of the surface must take the form

$$u_1 + u_1 v_1 + u_3 + \dots \dots + u_n = 0,$$

with the usual meaning for the suffixes. Taking the tangent plane as $z = 0$ and the normal as the z-axis, consider the general equation of the surface in the form of an infinite series

$$z = \frac{1}{2!}(rx^2 + 2sxy + ty^2) + \frac{1}{3!}(\alpha x^3 + 3\beta x^2 y + 3\gamma xy^2 + \delta y^3) + \dots$$

* The converse is not true. At a double point, we need not have necessarily L = 0, M = 0, N = 0. I have not been able to find the *necessary* conditions for a double point in terms of the fundamental magnitudes.

When $L\,dp^2 + 2M\,dp\,dq + N\,dq^2 = 0$ becomes an identity at the origin, so must $rx^2 + 2sxy + ty^2 = 0$, since either equation gives the inflexional directions at the point. The partial differential co-efficients of z of the second order therefore all vanish.

We now write down the following properties:

(1) The section of the surface made by the tangent plane at O is a plane curve with a triple point at O.

(2) There are three straight lines having contact of the third order with the surface at O.

(3) O is a point of inflexion for every normal section through O.

(4) There are three principal directions at O, *i.e.*, directions such that the normal at O meets the consecutive normal.

(5) The point O provides a case wherein the condition $LN - M^2 = 0$, or its equivalent $rt - s^2 = 0$ is not sufficient for a parabolic point. Nevertheless, O lies on the Hessian.

(6) We must have at O:

$$\begin{vmatrix} x_{11} & y_{11} & z_{11} \\ x_{12} & y_{12} & z_{12} \\ x_{22} & y_{22} & z_{22} \end{vmatrix} = 0$$

$$\left(x_{11} = \frac{\partial^2 x}{\partial p^2},\ x_{12} = \frac{\partial^2 x}{\partial p \partial q},\ \text{etc.} \right)$$

This follows from the relations

$$L = Xx_{11} + Yy_{11} + Zz_{11},\ \text{etc.}$$

It should be noted that the intersection of a surface with its Hessian includes besides the parabolic curve and the multiple lines, the locus of points with cubical indicatrices, whenever such a locus exists.

The conditions $\frac{L}{E} = \frac{M}{F} = \frac{N}{G}$

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or their equivalents $\frac{r}{1+p^2} = \frac{s}{pq} = \frac{t}{1+q^2}$

are usually regarded as the necessary and sufficient conditions for an umbilic. Exception should be made of the points at which L, M, N or r, s, t, all vanish.

2. Taking the equations to the edge of regression of a developable as $x = f_1(p)$, $y = f_2(p)$, $z = f_3(p)$, the equations of the developable can be written

$$x = f_1(p) + qf'_1(p)$$
$$y = f_2(p) + qf'_2(p)$$
$$z = f_3(p) + qf'_3(p).$$

It is verified that M = 0, and N = 0 identically. The value of L is given by

$$\frac{q\begin{vmatrix} f'_1(p) & f'_2(p) & f'_3(p) \\ f''_1(p) & f''_2(p) & f''_3(p) \\ f'''_1(p) & f'''_2(p) & f'''_3(p) \end{vmatrix}}{\{(f'^2_1 + f'^2_2 + f'^2_3)(f''^2_1 + f''^2_2 + f''^2_3) - (f'_1f''_1 + f'_2f''_2 + f'_3f''_3)^2\}^{\frac{1}{2}}}$$

$= q\psi(p)$, say.

(i) The equation $q = 0$ gives the edge of regression which is known to be a double line. The following will be a simple geometrical proof of this fact.* Let A and B be two consecutive generators meeting at the point P on the edge of regression. Then the tangent plane at P to the surface will be that corresponding to the generator A as well as that

* Another interesting proof is as follows: It is well-known that if the asymptotic lines of a surface possess an envelope, this envelope must be a plane curve, since it is simultaneously an asymptotic line and a line of curvature. [*Vide* Goursat and Hedrick: ***Mathematical Analysis***, Vol. II, Part II, p. 206, Example 6. Also ***Math. Annalen***, 99 Band, p 274.] If therefore the asymptotic lines of a surface possess a skew curve as the envelope, this skew curve can be an asymptotic line as well as a line of curvature ***only in a sense, viz.***, when the curve is a double line.

corresponding to the generator B. (A developable has the same tangent plane at all points of a given generator). There can only be one tangent plane at an ordinary point on a surface. If there are two tangent planes at any point, we can conclude that there are an infinite number of tangent planes and hence the point must be a double point.

(ii) Let $p = k$ be a root of $\psi(p) = 0$. The generator $p = k$ (which, by the way, meets the edge of regression at a point of zero torsion) is not necessarily a double line. It is therefore in general a locus of cubical indicatrices. One particular case wherein we obtain a double line is worth noting: *viz.*, when the point $x = f_1(k)$, $y = f_2(k)$, $z = f_3(k)$ is a point of inflexion on the edge of regression. The two equations determining X, Y, Z the direction cosines of the normal to the surface, *viz.*,

$$Xx_1 + Yy_1 + Zz_1 = 0,$$

$$Xx_2 + Yy_2 + Zz_2 = 0,$$

become identical if $f'_1 : f'_2 : f'_3 = f''_1 : f''_2 : f''_3$ which are the conditions for a point of inflexion on the cuspidal edge. X, Y, Z are therefore indeterminate at any point along a stationary generator. Hence,

a stationary generator (if any exist) on a developable must be a double line.

(iii) The developable furnishes an example wherein the conditions $L = 0$, $M = 0$, $N = 0$ are not necessary for a double point.* For it is well-known that every developable contains a nodal curve which is the locus of the points of intersection of non-consecutive generators. The works of Cayley and of Salmon contain detailed information about this result with examples.

---

* The question naturally arises: At such a double point, what does the equation $Ldp^2 + 2Mdpdq + Ndq^2 = 0$ represent? What does the line (X, Y, Z) represent? In all probability, it is the normal to one of the two 'tangent planes' at the point, and $L\,dp^2 + 2M\,dp\,dq + N\,dq^2 = 0$, the 'hyper-inflexional' directions in this plane. This would mean that the point is necessarily a binode or a unode, the case of the conic node being excluded.

Example (1). The developable surface

$$x = (p^4 - 2) + 4p^3q;\ y = (p^2 + 1) + 2pq;\ z = (p - 1) + q$$

has the Cartesian equation

$$16(z + 1)^2 \{(x + 2) - (y - 1)^2\} \{(y - 1) - (z + 1)^2\}$$
$$= \{(x + 2) - 4(y - 1)(z + 1)^2 + 3(y - 1)^2\}^2$$

The edge of regression given by

$$x + 2 = (y - 1)^2;$$
$$y - 1 = (z + 1)^2$$

is a double line. One of the factors of L is p, but the corresponding generator $x + 2 = 0;\ y - 1 = 0$ is not a double line but a locus of cubical indicatrices.

(2) The line $z = 0;\ w = 0$ is a stationary generator on the envelope of the planes

$$xt^4 + 4yt^3 + 6\lambda wt^2 + 4zt + w = 0,$$

where $x = 0,\ y = 0,\ z = 0,\ w = 0$ denote the equations of any four planes*. Taking the first three as the co-ordinate planes and $w = 0$ as any other plane, it is evident that

$$z = 0;\ w = 0$$

is a double line on the developable whose equation is

$$(xw - 4yz + 3\lambda^2w^2)^3 = 27(\lambda xw^2 + 2\lambda yzw - xz^2 - wy^2 - \lambda^3w^3)^2.$$

3. The method of curvilinear co-ordinates for discussing the above problems is believed to be new. But the theorem that the edge of regression of a developable is a double line is well-known though the usual proof that is given is more or less experimental.† More generally, the name 'cuspidal edge' given to the envelope of the characteristic curves on the singly infinite system of surfaces $f(x, y, z, a) = 0$ suggests that it is a locus of unodes or cusps on the enveloping surface.

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* Salmon : *Analytical Geometry of Three Dimensions* § 324.

† Salmon : *l. c.*, Art. 323, foot-note.

The only rigorous proof, however, that I have come across of this general proposition is in a paper by Bromwich and Hudson* who prove it by using the Newton-Puiseaux diagram. In this article is considered a somewhat different proof for the case of the developable.

To prove that the edge of regression of a developable is a locus of its unodes.

It is a theorem which was enunciated by M. J. M. Hill that if a surface possesses a continuous locus of double points, the double points should be binodes or unodes. †

Now consider any point on the edge of regression. This point must at least be a binode. The two generators which meet at the point are two among the infinite number of inflexional tangents which pass through that point. These generators being consecutive lines (which include an infinitesimal angle between them) must lie along the same nodal tangent plane. In other words, the osculating plane at any point on the edge of regression is a nodal tangent plane.

Choosing a point on the edge of regression as origin, the tangent, the principal normal and the binormal as the axes of co-ordinates, the equation of the surface should therefore be of the form

$$f(x, y, z) \equiv z(lx + my + nz) + \text{(terms of the third, fourth, etc., degrees)} = 0.$$

$z = 0$ being the osculating plane to the edge at the origin must touch the surface all along the generator $y = 0$; $z = 0$. The equation therefore should have the form

$$f \equiv z(lx + my + nz) + (y^2 v_1 + z u_2) + (y^2 v_2 + z u_3) + \dots$$

where suffixes denote the dimensions of the expressions.

---

* *Quarterly Journal of Mathematics* (1903).

† Art. 11 of his memoir in the *Philosophical Transactions of the Royal Society of London.* A (1892). The theorem has however a close bearing with some general theorems of Segre (*Annali di Mathematica* Series II, Vol. XXV, p. 28; Basset: *Geometry of Surfaces*, Secs. 24-25).

If $(x, 0, 0)$ denotes a consecutive point on the edge of regression, this being a double point on the surface, we must have $\frac{\partial f}{\partial z} = 0$. Hence $l = 0$.

The section of the surface by the normal plane $x = 0$ is given by $x = 0$; $z(my + nz) + \text{etc.} = 0$. Now this section should in the neighbourhood of the origin be in the form of a semi-cubical parabola with the principal normal $x = 0$; $z = 0$ as the cuspidal tangent*. Hence $m = 0$. The equation of the surface therefore reduces to the form

$$f \equiv z^2 + v_3 + v_4 + \dots \dots = 0.$$

The origin is therefore a unode. Hence the theorem.

Corollary 1. Let $f(x, y, z) = 0$ denote the equation (rationalised) of a developable surface other than a cone or cylinder, and let a, b, c, f, g, h denote the expressions

$$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial z^2}, \frac{\partial^2 f}{\partial y \partial z}, \frac{\partial^2 f}{\partial z \partial x}, \frac{\partial^2 f}{\partial x \partial y}.$$

The co-ordinates of any point on the edge of regression satisfy the following equations:

$$f = 0; \frac{\partial f}{\partial x} = 0; \frac{\partial f}{\partial y} = 0;$$

$\frac{\partial f}{\partial z} = 0$; every first minor of the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

equals zero.

If (x_0, y_0, z_0) is a point on the edge of regression of a developable $f(x, y, z) = 0$, the equation of the osculating plane to the edge of regression is

$$a_0 (x - x_0) + h_0 (y - y_0) + g_0 (z - z_0) = 0.$$

---

* *Vide* Eisenhart : *Differential Geometry*, Art. 20.

Corollary 2. *The edge of regression of the developable surface* $f(x, y, z) = 0$ *is an asymptotic line on each of the surfaces*

$$\frac{\partial f}{\partial x} = 0; \quad \frac{\partial f}{\partial y} = 0; \quad \frac{\partial f}{\partial z} = 0.$$

Corollary 3. A stationary point on a developable is a point where three consecutive generators may be supposed to intersect. At such a point, the plane through any two of the three generators must be a unodal plane, which is therefore indeterminate. The same fact can also be seen otherwise, using a more rigorous language. A stationary point is a cusp on the edge of regression (*vide* Salmon: *loc. cit.*) and the osculating plane thereat is thus indeterminate. It follows from Corollary 1, that at such a point a, b, c, f, g, h are all zero. Hence, *stationary point must be a multiple point on the surface of order higher than two.*

A similar argument can be made for the point where a stationary generator meets the edge of regression.

Conversely, *if the developable be constructed having a given curve as its edge of regression, the cusps and points of inflexion on the edge will be triple points on the surface.*

Example. If $w = 0$ denotes any plane, the point $x = y = z = w$ is a stationary point on the developable enveloped by the planes $xt^4 + 4yt^3 + 6zt^2 + (4t + 1)w = 0$ (Salmon). Forming the t-discriminant $f(x, y, z) \equiv I^3 - 27 J^2 = 0$, it is verified that the point is a triple point on the developable.

VIBRATIONS OF AN ELLIPTIC PLATE.

By J. Ghosh, M.A., Ph.D., *University of Dacca.*

1. Introduction.

The problem of the vibrations of an elliptic plate was first discussed by Mathieu.* But the theory of Mathieu functions has been developed not very long ago.† A form of solution of Mathieu's equation suitable for certain porposes was given by Whittaker.‡ The writer worked out the problem of the vibrations of an elliptic plate with the help of Whittaker's solution.§ In the present paper the writer has presented the solution as well as the frequency equation in a more general form.

2. The Equation of Motion.

The equation of the transverse vibration of an isotropic plate is‖

$$\nabla^4 w + \frac{3\rho(1-\sigma^2)}{E\tau^2}\frac{\partial^2 w}{\partial t^2} = 0. \qquad \dots \quad (2.1)$$

where 2τ = the thickness of the plate. Assuming w is a constant multiple of $\cos(pt+\varepsilon)$, we have

$$(\nabla^4 - k^4)\, w = 0, \quad k^4 = \frac{p^2}{c^2}, \quad c^2 = \frac{E\tau^2}{3\rho(1-\sigma^2)} \qquad \dots \quad (2.2)$$

We may write the equation as

$$(\nabla^2 + k^2)(\nabla^2 - k^2)\, w = 0 \qquad \dots \quad (2.3)$$

and if w_1 and w_2 are the solutions of $(\nabla^2+k^2)\,w_1=0$ and $(\Delta^2-k^2)\,w_2=0$ respectively, then $w = Aw_1 + Bw_2$ will be *a solution* of (2.3).

---

* *Liouville*, t. xiii, 1868.

† The study of Mathieu functions practically began with Prof. Whittaker's Report on the subject before the British Association in 1912.

‡ *Proc. Edin. Math. Soc.*, Vol. 32 (1913-14).

§ *Proc. Edin. Math. Soc.*, Vol. 44 (1925-26).

‖ Rayleigh: *Theory of Sound*, Vol. I. Chap. X.

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Applying the elliptic transformation $x + iy = h\cos(\xi + i\eta)$, the equation $(\nabla^2 + k^2)\, w_1 = 0$ reduces to

$$\frac{\partial^2 w_1}{\partial \xi^2} + \frac{\partial^2 w_1}{\partial \eta^2} + \frac{h^2k^2}{2}(\cosh 2\xi - \cos 2\eta)\, w_1 = 0 \quad \ldots \quad (2.4)$$

Assuming $w_1 = F_1(\xi)\, G_1(\eta)$, where $F_1(\xi)$ is a function of ξ only, and $G_1(\eta)$ is a function of η only, the equation breaks up into the two equations:

$$\frac{\partial^2 F_1}{\partial \xi^2} - (4a - 16q\cosh 2\xi)\, F_1 = 0 \quad \ldots \quad (2.5)$$

$$\frac{\partial^2 G_1}{\partial \eta^2} + (4a - 16q\cos 2\eta)\, G_1 = 0 \quad \ldots \quad (2.6)$$

where $4a$ is an arbitrary constant and $q = \frac{1}{32} h^2k^2$.

Similarly, the equation $(\nabla^2 - k^2)\, w_2 = 0$ will break up into two equations in $F_2(\xi)$ and $G_2(\eta)$ which differ from (2.5) and (2.6) only in the sign of q.

3. The Solution.

The solutions of the equation (2.5) are $Ce_0(\xi)$, $Ce_1(\xi)$, ... and $Se_1(\xi)$, $Se_3(\xi)$, ...

The solutions of the equation (2.6) are $ce_1(\eta)$, $ce_1(\eta)$, ... and $se_1(\eta)$, $se_2(\eta)$, ...

The solution of the equation (2.4) may therefore be written as

$$w_1 = A_0 Ce_0(\xi)\, ce_0(\eta) + A_1 Ce_1(\xi)\, ce_1(\eta) + \ldots \ldots$$
$$+ B_1 Se_1(\xi)\, se_0(\eta) + B_2 Se_2(\xi)\, se_2(\eta) + \ldots \ldots \quad (3.1)$$

where A_0, B_1, ... are arbitrary constants.

Let $\bar{C}e(\xi)$, $\bar{c}e(\eta)$, $\bar{S}e(\xi)$, $\bar{s}e(\eta)$ be the values of $Ce(\xi)$, $ce(\eta)$, $Se(\xi)$, $se(\eta)$ when $-q$ is substituted for q. Then the solution of the equation $(\nabla^2 - k^2)\, \omega^2 = 0$ may be written

$$\omega_2 = L_0 \bar{C}e_0(\xi)\, \bar{c}e_0(\eta) + L_1 \bar{C}e_1(\xi)\, \bar{c}e_1(\eta) + \ldots \ldots$$
$$+ M_1 \bar{S}e_1(\xi)\, \bar{s}e_1(\eta) + M_2 \bar{S}e_2(\xi)\, \bar{s}e_2(\eta) + \ldots \ldots$$

where L_0, M_1, etc., are arbitrary constants.

Hence the solution of the equation (2.1) may be written

$$w = \cos(pt + \varepsilon)$$

$$[A_0 Ce_0(\xi)\, ce_0(\eta) + L_0 \bar{C}e_0(\xi)\, \bar{c}e_0(\eta)$$

$$+ A_1 Ce_1(\xi)\, ce_1(\eta) + L_1 \bar{C}e_1(\xi)\, \bar{c}e_1(\eta)$$

$$+ B_1 Se_1(\xi)\, se_1(\eta) + M_1 \bar{S}e_1(\xi)\, \bar{s}e_1(\eta)$$

$$+ B_2 Se_2(\xi)\, se_2(\eta) + M_2 \bar{S}e_2(\xi)\, \bar{s}e_2(\eta) + \dots\dots] \quad \dots (33)$$

4. The Frequency Equation.

We shall assume that the plate is clamped at the boundary $\xi = \xi_0$ The major axis of the ellipse is $2h \cosh \xi_0$, the eccentricity is $(\cosh \xi_0)^{-1}$ and the focal distance $2h$. The conditions to be satisfied at the clamped edge are

$$w = 0, \quad \frac{\partial w}{\partial n} = 0. \quad \dots (4.1)$$

These are obviously equivalent to

$$w = 0, \quad \frac{\partial w}{\partial \xi} = 0. \quad \dots (4.1)$$

The vibrations fall into two classes, which we may distinguish as the C-class and the S-class.

We may then assume the solution

$$w = [A_n Ce_n(\xi)\, ce_n(\eta) + L_n \bar{C}e_n(\xi)\, \bar{c}e_n(\eta)] \cos(pt + \varepsilon)$$

as the typical normal mode of the C-class.

The boundary conditions (4.1) give

$$\left.\begin{array}{l} A_n Ce_n(\xi_0)\, ce_n(\eta) + L_n \bar{C}e_n(\xi_0)\, \bar{c}e_n(\eta) = 0; \\ A_n C'e_n(\xi_0)\, ce_n(\eta) + L_n \bar{C}'e_n(\xi_0)\, \bar{c}e_n(\eta) = 0. \end{array}\right\}$$

whence

$$\frac{Ce_n\ (\xi_0)}{C'e_n(\xi_0)} = \frac{\overline{C}e_n\ (\xi_0)}{\overline{C}'e_n\ (\xi_0)} \qquad \dots \quad (4.2)$$

which is the typical frequency equation of the C-class.

Similarly, the typical normal mode of the S-class is given by

$$w\ [B_1 Se_n\ (\xi)\ se_n\ (\eta) + M_1 \overline{S}e_n\ (\xi)\ \overline{se}_n(\eta)]\ \cos(pt + \varepsilon)$$

and the corresponding frequency-equation is

$$\frac{Se_n\ (\xi_0)}{S'e_n(\xi_0)} = \frac{\overline{S}e_n\ (\xi_0)}{\overline{S}'e_n\ (\xi_0)} \qquad \dots \quad (4.3)$$

The equations (4.2) or (4.3) give q, and then the corresponding frequency is obtained by means of (2.7) and (2.2). It should be observed, however, that the Mathieu functions have not yet been tabulated for any large number of values of n, but the cases $n = 0$. $n = 1$ and $n = 2$ may be worked out with the help of the tables given by Goldstein.

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* *Trans. Camb. Phil. Soc.* Vol. 23, No 11, (1927).

CAUCHY'S CRITERION FOR THE SOLVABILITY OF $x^p + y^p = z^p$ IN INTEGERS PRIME TO p.

By S. D. Chowla.

1. Cauchy stated, and it is not yet proved, that if the equation $x^p + y^p = z^p$ has any solutions in integers all prime to p, then

$$\sum_{n=1}^{\theta} n^{p-4} \equiv 0 \pmod{p} \qquad \dots \quad (1.1)$$

where $\theta = [\tfrac{1}{2} p]$; $[x]$ denotes the greatest integer contained in x.

In this note I show that this criterion is equivalent to

$$\sum_{n=1}^{\phi} n^{p-4} \equiv 0 \pmod{p} \qquad \dots \quad (1.2)$$

where $$\phi = [\tfrac{1}{4} p].$$

2. If m is an odd positive integer, it is easy to prove* that

$$T_n \equiv (-1)^n \, 2^{2n-1} \{ (m-1)^{2n-1} - (m-2)^{2n-1} + - \dots + 2^{2n-1} - 1^{2n-1} \} \pmod{m^2} \quad \dots \quad (2.1)$$

where T_n is the n'th prepared Bernoullian number. Hence if $T_n \equiv 0 \pmod{m}$, then

$$1^{2n-1} - 2^{2n-1} + - \dots - (m-1)^{2n-1} \equiv 0 \pmod{m}$$

whence

$$1^{2n-1} - 2^{2n-1} + - \dots \pm \left(\frac{m-1}{2}\right)^{2n-1} \equiv 0 \pmod{m} \quad \dots \quad (2.2)$$

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* *Messenger of Maths.*, Vol. 57. p. 121;

p denotes a prime in this paper.

Again, since m is odd

$$(m-1)^{2n-1} - + \dots + 2^{2n-1} - 1^{2n-1}$$

$$= 1^{2n-1} + 2^{2n-1} + \dots + (m-1)^{2n-1}$$

$$- 2.2^{2n-1}\left\{ 1^{2n-1} + 2^{2n-1} + \dots + \left(\frac{m-1}{2}\right)^{2n-1} \right\}$$

$$\equiv -2^{2n}\left\{ \sum_{S=1}^{\frac{1}{2}m} S^{2n-1} \right\} \pmod{m} \quad \dots \quad (2.3)$$

From (2.1) and (2.3) it follows that $T_n \equiv 0 \pmod{m}$, m odd, is equivalent to

$$\sum_{S=1}^{\frac{1}{2}m} S^{2n-1} \equiv 0 \pmod{m} \quad \dots \quad (2.4)$$

By (2.2), $T_n \equiv 0 \pmod{m}$, m odd, is equivalent to

$$\sum_{S=1}^{\frac{1}{2}m} (-1)^S S^{2n-1} \equiv 0 \pmod{m} \quad \dots \quad (2.5)$$

From (2.4) and (2.5) it follows that

$$\sum_{1}^{\frac{1}{2}m} S^{2n-1} \equiv 0 \pmod{m} \quad \dots \quad (2.6)$$

is the condition that T_n should be a multiple of the odd number m.

3. From (2.4) we see that Cauchy's criterion is equivalent to

$$T_{\frac{1}{4}(p-3)} \equiv 0 \pmod{p}$$

Hence from (2.6) Cauchy's criterion is equivalent to

$$\sum_{S=1}^{\frac{1}{4}p} S^{p-4} \equiv 0 \pmod{p} \quad \dots \quad (3.1)$$

which is half the length of Cauchy's criterion.

A NOTE ON A THEOREM OF HARDY*

By K. S. K. Iyengar.
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The object of this note is to give a proof by direct application of the Schwarzian inequality without any intermediary analysis, and different from Hardy's proof of a theorem given in *Mathematische Zeitshrift* (1920, Band 6, page 314), which states that, if $k > 1$, and $\sum^{\infty} a_n^k$ convergent, then

$$\sum^{\infty} \left(\frac{S_n}{n}\right)^k \leqslant \lambda . \sum^{\infty} a_n^k,$$

where $S_n = a_1 + \ldots + a_n$ and λ some constant.

Recently I have been able to get by the same method the best possible value for λ known to be $= \left(\frac{k}{k-1}\right)^k$.

1. Let $$0 < \alpha < k - 1.$$

Then by Schwarz's inequality

$$S_n = \sum_1^n a_n = \sum_1^n \frac{a_n . n^{\alpha/k}}{n^{\alpha/k}} . \leqslant \left(\sum_1^n a_n^k . n^{\alpha}\right)^{\frac{1}{k}} . \left(\sum_1^n \frac{1}{n^{\alpha/(k-1)}}\right)^{\frac{k-1}{k}}$$

and
$$\sum_1^n \frac{1}{n^{\alpha/(k-1)}} \leqslant \frac{1}{1-\alpha/(k-1)} . n^{1-\frac{\alpha}{(k-1)}}.$$

We therefore have

$$\left(\frac{S_n}{n}\right)^k \leqslant \lambda \left(\sum_1^n a_n^k . n^{\alpha}\right) . \frac{1}{n^{1+\alpha}}.$$

$$\therefore \sum^{\infty} \left(\frac{S_n}{n}\right)^k \leqslant \lambda . \sum^{\infty} a_n^k . n^{\alpha} . \left(\frac{1}{n^{1+\alpha}} + \frac{1}{(n+1)^{1+\alpha}} + \ldots\right)$$

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* Read at the Nagpur Conference of the Indian Mathematical Society (1928).

where $$\lambda = \left\{ \frac{1}{1-a/(k-1)} \right\}^{k-1}$$

$$\frac{1}{n^{1+\alpha}} + \frac{1}{(n+1)^{1+\alpha}} + \ldots \leqslant \frac{1}{n^{\alpha}} \left(\frac{1}{1^{\alpha+1}} + \frac{1}{2^{\alpha+1}} + + \frac{1}{r^{\alpha+1}} + \right).$$

We therefore have

$$\sum^{\infty} \left(\frac{S_n}{n}\right)^k \leqslant \frac{\xi(1+a)}{(1-a/k-1)^{k-1}} \cdot \left(\sum^{\infty} a_n^k\right)$$

In the following note is obtained the best possible form for λ by the same method. [Refer *Proc. Lond. Math. Soc.*, Vol. 25, page 274].

2. Let $u_{n+1} = u_1 \left(1 - \frac{a}{1+a}\right) \ldots \ldots \left(1 - \frac{a}{n+a}\right)$ ... (A)

where $a = 1 - \frac{1}{k}$ and $u_1 > 0$ and let c_n be so chosen that

$$\left(\sum \frac{1}{c_n^{1/k-1}}\right) = n^k \cdot \left(u_n - u_{n+1}\right). \quad \ldots \text{(B)}$$

It is easy to see that u_n decreases.

Then as before (in 1)

$$S_n = \sum_1^n a_n = \sum_1^n \frac{a_n \cdot c_n^{1/k}}{c_n^{1/k}} \leqslant \left(\sum_1^n a_n^k \cdot c_n\right)^{1/k} \cdot \left(\sum_1^n \frac{1}{c_n^{1/k-1}}\right)^{\frac{k-1}{k}}$$

$$\therefore \left(\frac{S_n}{n}\right)^k \leqslant \left(\sum_1^n a_n^k \cdot c_n\right) \cdot \left(u_n - u_{n+1}\right) \text{ by B.}$$

$$\therefore \sum^{\infty} \left(\frac{S_n}{n}\right)^k \leqslant \sum^{\infty} a_n^k \cdot c_n \cdot u_n.$$

We will now prove that:

$$u_n \cdot c_n \leqslant \left(\frac{k}{k-1}\right)^k,$$

which means the same as

$$\left(\frac{1}{c_n}\right)^{\frac{1}{k-1}} \geqslant \left(\frac{k-1}{k}\right)^{\frac{k}{k-1}} . u_n^{\frac{1}{k-1}}. \qquad \ldots \text{(C)}$$

Now

$$\frac{1}{c_n^{1/k-1}} = n^{k/(k-1)}(u_n - u_{n+1})^{1/(k-1)} - (n-1)^{k/(k-1)}(u_{n-1} - u_n)^{1/(k-1)}.$$

By A, $\quad u_n - u_{n+1} = \dfrac{u_n . a}{n+a}$;

$$u_{n-1} - u_n = u_n \left\{ \frac{1}{1 - a/(n-1+a)} - 1 \right\} = \frac{u_n . a}{n-1}.$$

$$\therefore \quad \frac{1}{c_n^{1/(k-1)}} = u_n^{1/(k-1)} . \left[\frac{n^{k/(k-1)}}{(n+a)^{1/(k-1)}} - (n-1) \right] a^{1/(k-1)}.$$

Since $a = 1 - \frac{1}{k}$, we have therefore to prove,

$$\left\{ \frac{n^{k/(k-1)}}{(n+a)^{1/(k-1)}} - (n-1) \right\} \geqslant a$$

i.e., $\quad n^{k/(k-1)} - (n+a)^{k/(k-1)} \geqslant -(n+a)^{k/(k-1)}$.

i.e., $\quad (n+a)^{k/(k-1)} - n^{k/(k-1)} \leqslant (n+a)^{1/(k-1)}$

i.e., $\quad \int_0^1 (n+ax)^{1/(k-1)} dx \leqslant (n+a)^{1/(k-1)}$

which is quite obvious.

3. By putting $a_n^k = b_n$ we have

$$\sum^{\infty} \left(\frac{b_1^{1/k} + b_2^{1/k} + \ldots + b_n^{1/k}}{n} \right)^k \leqslant \left(\frac{k}{k-1} \right)^k . \sum^{\infty} b_n$$

Let $k \to \infty$, then we have

$$\sum^{\infty} (b_1 . b_2 . b_n)^{\frac{1}{n}} \leqslant e . \sum^{\infty} b_n.$$

The latter theorem is due to Carlemann. It has been proved by Polya that the best possible constant on the right-hand side is e,

KAR'S THEORY OF INTERMITTENT ACTION AND THE BOWED STRING.

By Baidya Nath Biswas, M.Sc.

The application of the theory of intermittent action as developed by Kar has met with success in several cases.* In the present paper an attempt has been made to extend the theory referred to above, to the dynamics of bowed strings.

The equation of motion of a vibrating stiring in normal co-ordinates with the co-efficient of damping is †

$$\ddot{\phi}_m + 2k\,\dot{\phi}_m + a^2\,\phi_m = \frac{2}{l\rho}\cdot\Phi_m \quad \ldots \quad (1)$$

where l is the length, ρ the mass per unit length of the string, and k the co-efficient of damping.

The solution of the equation when $\Phi_m = 0$ is:

$$(\phi_m)_t = e^{-kt}\left\{\frac{(\dot{\phi}_m)_0}{P_m}\cdot\sin P_m t + (\phi_m)_0(\cos P_m t + \frac{K}{P_m}\sin P_m t)\right\} \quad \ldots \quad (2)$$

If the force Φ_m acts at $t = t'$ continuously throughout a small interval τ_0, then $(\phi_m)_t$ at any time is given by

$$(\phi_m)_t = \frac{2}{l\rho}\,\frac{1}{P_m}\int_{t'}^{t'+\tau_0}\Phi_m\cdot e^{-k(t-t')}\cdot\sin P_m(t-t')\cdot dt' \quad \ldots \quad (3)$$

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* K. C. Kar: *Phys. Rev.* 695, 1923: *Phys.* ZS. 24, 63, (1923).
K. C. Kar and M. Ghose, *Phys. Zeit* 29, 143, (1928); M. Ghose, *Phys. Zeit* 30, 160, (1929).
K. C. Kar. and R. C. Mazumdar, *Zeit. f. Phys.* Bd. 53, 308, (1929).
Kulesh Chandra Kar, *Zeit f. Phys.* Bd. 57, 416, (1929).
Kulesh Chandra Kar and Baidya Nath Biswas, *Zeit f. Phys.* Bd. 59, 570, (1930).

† Rayleigh: *Theory of Sound*, 2nd Ed., Vol. I, p. 74.

In the case of a bowed string Φ_m will be of the form

$$A_m \sin \frac{m\pi b}{l} \times \cos(P_m t - a_m),$$

b being the distance of the bowed point from one end.

Hence we have from equation (3)

$$(\phi_m)_t =$$

$$\frac{2}{l\rho}\frac{A_m}{P_m}.\sin\frac{m\pi b}{l}.e^{-kt}\int_{t'}^{t'+\tau_0} e^{kt'}.\cos(P_m t' - a_m).\sin P_m (t-t')\,dt'. \quad \ldots \quad (4)$$

On integrating we get

$$(\phi_m)_t = M.e^{-k(t-t')} \times$$

$$\Big\{ \frac{e^{k\tau_0}-1}{k}.\sin(P_m t - a_m) - e^{k\tau_0}.G.\sin(2P_m.\overline{t'+\tau_0} - a_m - P_m t)$$

$$+ S.e^{k\tau_0}.\cos(2P_m\overline{t'+\tau_0} - a_m - P_m t) - G.\sin(a_m + P_m t - 2P_m t')$$

$$- S\cos(a_m + P_m t - 2P_m t')\Big\} \quad \ldots \quad (5)$$

where $M = \frac{1}{l\rho}.\frac{A_m}{P_m}.\sin\frac{m\pi b}{l}$, $G = \frac{K}{K^2+4P_m^2}$, $S = \frac{2P_m}{K^2+4P_m^2}$.

We shall suppose that the force Φ_m instead of acting continuously throughout, acts continuously during the intervals zero to τ_0, τ to $\tau+\tau_0$, 2τ to $2\tau+\tau_0$, etc., where $\tau_0 < \tau$ (period).

If the force acts only from 0 to τ_0, we have for the value of $(\phi_m)_{(n+e)\tau}$, ($e < 1$) by putting $t = (n+e)\tau$, $t' = 0$ in equation (5).

$$(\phi_m)_{(n+e)\tau} = M.e^{-k(n+e)\tau} \times$$

$$\Big\{ \frac{e^{k\tau_0}-1}{k}\sin(P_m T - a_m) - e^{k\tau_0}.G.\sin(2P_m\tau_0 - a_m - P_m T)$$

$$+ S.e^{k\tau_0}\cos(2P_m\tau_0 - a_m - P_m T) - G.\sin(a_m + P_m T)$$

$$- S.\cos(a_m + P_m T)\Big\} \quad \ldots \quad (6)$$

where $T = e\tau$.

Similarly when the force acts from τ to $\tau + \tau_0$, we have

$$(\phi_m)_{(m+e)\tau} = M.e^{-k(n-1+e)\tau}$$

$$\left\{ \frac{e^{k\tau_0}-1}{\kappa}.\sin(P_m T - a_m) - e^{k\tau_0}.G.\sin(2P_m\tau_0 - a_m - P_m T)\right.$$

$$+ S.e^{k\tau_0}\cos(2P_m\tau_0 - a_m - P_m T) - G.\sin(P_m T + a_m$$

$$\left. + S\cos(P_m T + a_m)\right\} \qquad \dots \quad (7)$$

Thus if the force acts discontinuosly as above for n periods, we have remembering $P_m\tau = 2\pi m$,

$$(\phi_m)_t = M.e^{-kT} \times$$

$$\left[e^{-kn\tau} + e^{-k(n-1)\tau} + \dots + e^{-k\tau}\right]\left[\frac{e^{k\tau_0}-1}{k}.\sin(P_m T - a_m)\right.$$

$$- e^{k\tau_0}.G.\sin(2P_m\tau_0 - a_m - P_m T) + S.e^{k\tau_0}\cos(2P_m\tau_0 - a_m - P_m T)$$

$$\left. - G.\sin(P_m T + a_m) + S.\cos(P_m T + a_m)\right] \qquad \dots \quad (8)$$

where n is great, $e^{-k.n\tau} < e^{-k\tau}$ and hence we have on summation.

$$(\phi_m)_t = M.\frac{e^{-kT}.e^{-k\tau}}{1-e^{-k\tau}} \times$$

$$\left\{ \frac{e^{k\tau_0}-1}{k}\sin(P_m T - a_m) - e^{k\tau_0}\, G\, \sin(2P_m\tau_0 - a_m - P_m T)\right.$$

$$+ S.e^{k\tau_0}.\cos(2P_m\tau_0 - a_m - P_m T) - G\sin(P_m T + a_m)$$

$$\left. + S\cos(P_m T + a_m)\right\} \qquad \dots \quad (9)$$

Now, as each term in the expression within the second bracket involves $\sin P_m T$, $\cos P_m T$, we can write equation (9) in the following simple form

$$(\phi_m)_t = B_m.\frac{e^{-kT}.e^{-k\tau}}{1-e^{-k\tau}}.\sin P_m(T + \psi_m)$$

where B_m and ψ_m are two arbitray constants.

As the vibration of the string is symmetrical about its position of rest, we have:

$$\phi_m = 0 \text{ at } T = \frac{\tau_0}{2}, \ \tau + \frac{\tau_0}{2}, \ 2\tau + \frac{\tau_0}{2}, \text{ etc.}$$

In order that equation (10) may satisfy the physical conditions of the problem we must have $\psi_m = -\frac{\tau_0}{2}$. ... (10)

Thus equation (10) becomes

$$(\phi_m)_t = \frac{B_m e^{-kT} e^{-k\tau}}{1 - e^{-k\tau}} \cdot \sin P_m \left(T - \frac{\tau_0}{2}\right) \qquad \text{... (11)}$$

Hence the displacement of the bowed point is given by

$$y = \sum_{m=1}^{m=\infty} B_m \cdot \frac{e^{-kT} \cdot e^{-k\tau}}{1 - e^{-k\tau}} \cdot \sin P_m \left(T - \frac{\tau_0}{2}\right) \qquad \text{... (12}$$

On differentiating with respect to the time, we have

$$\dot{y} = \sum_{m=1}^{m=\infty} B_m \cdot \frac{e^{-kT} \cdot e^{-k\tau}}{1 - e^{-k\tau}} \cdot P_m \cos P_m \left(T - \frac{\tau_0}{2}\right)$$

$$-B_m \cdot \frac{k \cdot e^{-kT} \cdot e^{-k\tau}}{1 - e^{-k\tau}} \cdot \sin P_m \left(T - \frac{\tau_0}{2}\right) \qquad \text{... (13)}$$

During the interval 0 to τ_0 the velocity of the bowed point is by Helmholtz second Law equal to the velocity V of the bow. During the nterval $\tau - \tau_0$ when there is slip between the bow and the string, $\dot{y}$ generally fluctuates within a small range and the mean velocity during this interval may be taken as $\frac{V \cdot \tau_0}{\tau - \tau_0}$ (displacement ÷ interval $\tau - \tau_0$).

Thus on multiplying the both sides of (13) by $\cos P_m \left(T - \frac{\tau_0}{2}\right)$ and integrating between the limits zero and τ we get, regarding e^{-kT} as practically constant and k being small,

$$V\int_0^{\tau_0}\cos P_m\left(T-\frac{\tau_0}{2}\right)dT-\int_{\tau_0}^{\tau}\frac{V.\tau_0}{\tau-\tau_0}.\cos P_m\left(T-\frac{\tau_0}{2}\right)dT$$

$$=B_m\frac{e^{-k\tau}.e^{k-\tau}}{1-e^{-k\tau}}.\frac{P_m\tau}{2} \qquad \ldots \quad (14)$$

On integrating the left-hand side of (14) becomes equal to

$$\frac{2\tau V\sin\frac{1}{2}P_m\tau_0}{P_m(\tau-\tau_0)}.$$

Hence putting $P_m=\dfrac{2m\pi}{\tau}$ we get :

$$B_m=\frac{V.\tau^2}{\pi^2.m^2\left(\tau-\frac{\tau_0}{2}\right)}.\frac{1-e^{-k\tau}}{e^{-2k\tau}}.\sin\frac{m\pi\tau_0}{\tau} \qquad \ldots \quad (15)$$

Therefore from (12) and (15) we have

$$y=\frac{V.\tau^2}{\pi^2.(\tau-\tau_0)}\sum_{m=1}^{m=\infty}\frac{e^{-kT}}{e^{-k\tau}}.\frac{1}{m^2}\sin\frac{m\pi\tau_0}{\tau}.\sin\frac{2m\pi}{\tau}\left(T-\frac{\tau_0}{2}\right).$$

$$\ldots \quad (16)$$

which is the well-known equation for a bowed string except for the factor $\dfrac{e^{-kT}}{e^{-k\tau}}$ which is evidently very nearly equal to unity.

CREMONA TRANSFORMATION WITH A CONIC OF FIXED POINTS.*

BY S. AUDINARAYANAN.

Introduction.

Doehlemann † and Castelnuovo ‡ have considered the subject of Cremona transformations with a curve of fixed points. Doehlemann has shown that if T of order n has a curve of fixed points of order μ, then $n \geqslant (3\mu - 1)/2$ and Castelnuovo proves that if T has an irreducible curve of fixed points of genus $p > 1$, then either T is periodic and of period 2, 3 or 4 or T is the transform of a Jonquiere's type. It is the purpose of this paper to consider the case of transformations of low orders with a conic of fixed points. I have also considered the general De Jonquiere's type with a conic of fixed points.

After a preliminary paragraph on definitions and notations, I consider the general equations of transformation with a conic of fixed points and find the conditions for the transformation being a Cremona transformation. In § 3 I consider the well-known quadratic transformation with a conic of fixed points and its projective treatment. After a resume of composition of general transformations, I pass on to the case of the Jonquiere's type with a fixed conic and express the general case as product of several T_2's each having the conic for a fixed conic. § 6 is devoted to the case of quartic transformation of the semi-symmetric type and the two cases in which they have a conic of fixed points are discussed and in both cases the transformation is shown to be a product of three T_2's with the same conic of fixed points. Finally, I consider a general quartic transformation defined by a conic.

1. Preliminary definitions and remarks.§

A Cremona plane transformation is a (1, 1) relation between the

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* I am greatly indebted to Dr. R. Vaidyanathaswamy for suggestions and criticisms in the preparation of this paper.

† *Math. Aun* Bd. 39, pp. 567—597.

‡ *Roma. Acc. Linei Rendioonti* (5) Vol 1, pp 47—50,

§ The notation and references are to Hudson's "*Cremona Transformations.*"

points P, P′ of two planes S, S′. If (x, y, z) are the co-ordinates of P, then the co-ordinates (x', y', z') of P′ are uniquely detetmined and *vice versa.* There exist sets of relations

$$T\ :\qquad x' : y' : z' : \equiv \phi_1 : \phi_2 : \phi_3, \qquad \dots \quad (1)$$

$$T^{-1} :\qquad x : y : z : \equiv \phi'_1 : \phi'_2 : \phi'_3, \qquad \dots \quad (2)$$

where ϕ_i, ϕ_i' are homogeneous polynomials in (x, y, z) and (x', y', z') respectively and have no common factors.

The net of curves,

$$(\phi) \equiv c_1 \phi_1 + c_2 \phi_2 + c_3 \phi_3$$

are such that any two curves have only one free intersection other than the base points of the net. The net (ϕ) is called the first homoloidal net of T and the base points of the net are called the F-points of the transformation. Similarly, the net (ϕ') is called the second homoloidal net and its base points are the second F-system. Each curve of the net is the transform of a straight line in the other plane. By considering the intersections of a line and a curve (ϕ) in S, and also the corresponding (ϕ') and the line in S′ it is easily seen that the curves (ϕ), (ϕ') are of the same degree.

*The F-points of the homoloidal net of curves are points of different multiplicities such that the curves are of genus zero and yet do not reduce the net to a pencil or a single curve. A Cremona transformation breaks down at the F-points, ϕ_1, ϕ_2, ϕ_3 all vanishing at once. The equations (1) cease to be significant and (2) reduce to a single equation representing a curve instead of a point. This curve is called a principal curve or a P-curve. The set of P-curves corresponding to the F-points of the first system is the second P-system. The P-curve corresponding to an i-ple F-point is of degree i and there are as many P-curves as F-points. There is also the first P-system corresponding to the second F-system.

If a curve f_μ passes through an i-ple F-point, its transform will consist of the corresponding P-curve and a curve of degree $\mu n - i$. When we consider the two planes S, S′ to be superposed* there will occur

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* Hudson: *loc. cit.*, p. 78.

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NOTES AND QUESTIONS.

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Notes and Questions.

Mathematics in the Fourteenth Edition of the Encyclopædia Britannica.

Students and teachers of mathematics have learnt by long and varied experiences to prize very highly some of the articles relating to their subject which have appeared in the various editions of the *Encyclopædia Britannica* and hence they are naturally greatly interested in the most recent edition of this valuable work. It may therefore be of interest to consider here briefly some of the questionable statements which appear in this edition in view of the fact that new light may thus be thrown on various points of general interest and that it may enable the reader to secure more easily an insight into the relative merits of this edition in comparison with the earlier ones It is not assumed that the reader will agree in every case that the views expressed here are better than those found in the work under consideration, but these supplementary views will doubtless enable him to form a more satisfactory judgment relating to the questions involved than if he had considered only one of these views. The mathematician naturally welcomes the fullest discussion since his subject grows in beauty as the details relating thereto are more fully revealed.

Under the term "Algebra," volume 1, page 607, there appears the following statement: "Elementary algebra may be said to have been substantially completed by the close of the 17th century, by which time, higher algebra, especially through the progress made in the study of equations, symmetric functions and series, had already begun to rank as a special branch of mathematics". This quotation may profitably be compared with the following from volume 2 (1921), page 78 of Tropfke's *Geschichte der Elementar-Mathematik:* "Das achtzehnte Jahrhundert krankt daran dass eine allgemein befriedigende Einführung der negativen Grössen fehlt. Die Zeichen + und — werden beim Rechnen verwendet und dann ganz unvermittelnd zur Bezeignung positiver und negativer Zahlen benutzt. Man beachtet nicht die Notwendigkeit, die Zulässigkeit das Rechnens mit ihnen, wie die Multiplicationgesetze, zu prüfen und zu beweisen."

It is obvious that a satisfactory introduction to negative numbers is a fundamental element of elementary algebra. Moreover, an introduction to the subjects of graphs, complex numbers, and determinants frequently appears in the elementary algebras of our times, and these introductions are usually largely based on developments which were made after the close of the 17th century. In particular, it is well-known that the elementary algebras of to-day owe much to the work of L. Euler, 1707—1783. The most objectionable feature of the quotation under consideration is its vagueness since the terms 'elementary algebra' and 'higher algebra' are used in our times with such widely different meanings. Historical statements frequently are almost of no value on account of the use of vague terms therein.

For instance, on the top of the page of the Britannica from which we quoted, there appears the statement: "The Chinese were able to solve the quadratic equation before the Christian era." If one recalls that irrational numbers and complex numbers are needed in the general solution of the quadratic equation, one can easily see that the means for the general solution of the quadratic equation were not available at the beginning of the Christian era. The following quotation from the same page of the Britannica may appear still more questionable to many readers: "Italy was the centre of learning and her scholars devoted much attention to solving the cubic equation. The latter solution was finally effected with substantial completeness, by Tartaglia, being published by Cardan in his *Ars Magna* (1545). The biquadratic was solved by Ferrari (1540) a pupil of Cardan's and was also published by the latter in the same work." Nothing is said here of the credit due to Ferro in regard to the solution of the cubic equation and the date which appears after the name of Ferrari is questionable.

It may be added that the work of Ferro relating to the solution of the cubic equation is regarded as so important by J. Tropfke that he calls the formula which is commonly called Cardan's Formula by the name of Ferro's Formula on page 75 of volume 3 (1922) of his well and favourably known *Geschichte der Elementar-Mathematik.* Moreover, on page 86 of the same volume, Tropfke directs attention to the fact that the highly-gifted pupil Ferrari could not have been 23 years old when he made the discovery of the solution of the biquadratic equation, since he was born in 1522. If the date 1540 given above is correct, he was still much

younger at the time of this fundamental solution. It should also be noted that it is questionable whether one can properly say that Tartaglia effected the solution of the cubic with substantial completeness since J. L. D'Alembert seems to have been the first to note that the formula in question gives the three roots of this equation and not only one root thereof. This was more than two hundred years after the publication of the noted *Ars Magna* by Cardan. There is a vast difference between using a general rule and comprehending its generality.

Under the term "Arithmetic," volume 2, page 356, there appear the following statements:—"The distinction between *abstract numbers*, like 4, and *concrete numbers* like 4 ft, is an inheritance that serves no important purpose," and "Denominate numbers in general and compound numbers in particular trace their origin to the difficulty which the ancients had in developing a satisfactory notation for either integers or fractions." It is obvious that the term number could be so defined as to apply only to what is now known as an abstract number but this question does not seem to be involved in these quotations. The existence of abstract numbers and concrete numbers is acknowledged here but it is said that the distinction between these numbers serves no important purpose. This remark is the more striking in view of the fact that stress is laid on the historical development of mathematics in the most recent edition of the Britannica and the development of the concept of abstract numbers is also of fundamental historical importance.

It seems as absurd to say that the distinction between abstract and concrete numbers serves no important purpose as it would be to say that thinking serves no important purpose in mathematics, and it seems reasonable to assume that denominate numbers would have been used by the ancients even if they had had a good notation for abstract numbers. It is true that compound numbers serve a purpose somewhat akin to that of local value in a numerical notation and it is possible that the use of compound numbers had some influence on the development of a numerical notation based on local value. There is, however, no historical evidence to indicate that denominate and compound numbers in arithmetic had their origin in the difficulties which the ancients experienced in adopting a good numerical notation. Hence the positive statement that these numbers trace their origin to these difficulties is quite misleading. The modern metric system was adopted long after a good numerical notation

was in use and various systems of compound numbers in the sciences have been of comparatively recent origin.

Under the term "Fraction," Volume 9, page 575, it is stated that the "ratio of 2 to 43 is expressed (to use modern symbols) as

$$\frac{1}{42}+\frac{1}{86}-\frac{1}{129}+\frac{1}{301}."$$

The fact that a negative sign appears here in the combination of unit fractions is apt to be quite misleading for the beginner since the unit fractions in the well known table of Ahmes are always combined by addition. Near the bottom of the same page there appears a much more significant historical statement as follows: "The Arabs introduced the bar, but they did not make use of it in all cases." This positive statement may be compared with the following: "Die heutige Schreibart eines Bruches geht wie im allgemeinen Ueberblick bemerkt ist, aus indischen hervor, die die Araber übernahmen. Der Zähler stand oberhalb des Nenners. Durch einen Strich (virgula) wurden beide erst im *Liber abaci* (1228) des Leonardo von Pisa voneinander getrennt" J. Tropfke, *Geschichte der Elementar-Mathematik*, Volume 1 (1921), page 128.

Under the term "Groups," Volume 10, page 914, appears the following remarkable statement: "For example, if the order of the group is

$$N = p_1^{n_1} p_2^{n_2} \dots \dots p_s^{n_s}$$

where the p_i are distinct primes, $A_1, A_2, \dots, A_s$ may be so selected that their orders are

$$a_1 = p_1^{n_1},\ a_2 = p_2^{n_2},\ \dots \dots,\ a_s = p_s^{n_s}$$

respectively." The symbols $A_1, A_2, \dots A_s$ in this quotation represent a base of the group and hence this quotation implies that every Abelian group is cyclic. As this relates to such a fundamental theorem in Abelian groups, it is quite likely to annoy the beginner even if he should be able to see from what follows that the theorems quoted contradict each other if they are actually supposed to represent general cases. The quotation actually states that an Abelian group can always be generated by a number of operators which need not exceed the number of the distinct

prime numbers which divide its order while the least number of such generators is the largest number of invariants of any of its Sylow subgroups, as is well-known to those who are familiar with the theory of Abelian groups.

There is, however, another statement in this same column which is perhaps more apt to give rise to misconceptions in regard to a fundamental matter. This statement is as follows: "An isomorphism of a group defines a substitution

$$S' = \begin{Bmatrix} 1, & S_2, & S_3, & \dots & S_n \\ 1, & S'_2, & S'_3, & \dots & S'_n \end{Bmatrix}$$

and the totality of such isomorphisms forms an intransitive substitution group L isomorphic to G." The fact that the group L is not necessarily isomorphic to G in the ordinary sense of this term follows directly from the non-cyclic group of order 4. In this case L is known to be of order 6 and to be simply isomorphic with the symmetric group of this order and not to the four-group. As the subject of isomorphisms is a relatively difficult one, this quotation is apt to cause more trouble to the reader than the preceding one.

We may refer here also to a historical statement relating to groups, even if it is not apt to affect so many readers as those already noted. This appears on the same page as the preceding ones and is as follows "Our knowledge of the structure of finite groups depends largely upon a set of closely related theorems of which Sylow's theorem was the first to be proven." It is well-known that the theorem commonly known as Lagrange's theorem was proven much earlier in the same sense as Sylow's theorem was proven in 1872. Both of these theorems and various others are generally said to have been proven before a general definition of an abstract group had been given and hence these proofs might be questioned. This is, however, not commonly done notwithstanding the fact that the theorems became much richer in content long after their proofs had become known. Great complications in historical questions would arise if credit for proofs was withheld, because the authors thereof were not in a position to foresee the enrichment of their theorems by the work of later generations, but when these proofs apply equally to these richer theorems.

Under the term "Logarithms," volume 14, page 305, there appears the following remark: "Natural logarithms first arose as more or less accidental variations of Napier's original logarithms. Their real significance in analysis was not recognized until later. The earliest natural logarithms occur in the 1618 edition of Edward Wright's translation into English of Napier's *Descriptio*". With this quotation the reader may profitably compare the following: "Zweitens ist auf Nik. Mercators *Logarithmotechnia*, London, 1668, zu verweisen die die Entdeckung der logarithmischen Reihe:

$$l(1+a) = a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \dots$$

enthielt und damit eine völlige Umwälzung in der Berechnung logarithmischer Tafeln anbahnte. Auch hier war P. Mengoli durch Aufstellung eigenartiger Reihen für die naturlichen Logarithmen (die er auch so nannte) der ganzen und gebrochenen Zahlen > 1 schon vorangegangen."

It seems very questionable whether the tables which were derived by modifying slightly the so-called Napier's tables of logarithms should be called natural logarithms, although this has often been done. While the tables thus derived agree with our natural logarithmic tables if only a few decimal places are used they would not agree therewith for a large number of decimal places. The theory which was employed in the construction of Napier's tables differs very widely from that involved in defining e, the base of our system of natural logarithms. Napier's work relating to the construction of tables was algebraic while it is a transcendental number and its approximate determination is based on an infinite series. Moreover, it is somewhat confusing to read about the existence of tables of natural logarithms long before the base of these logarithms had been defined. One cannot determine the base of a system of logarithm from a table giving only a certain number of decimals of the logarithms concerned. An infinite number of different bases would give rise to the same such table.

Under the term "Taylor's Theorem," Volume 21, page 856, one may find the following sentence:—"A related formula published by the Scottish mathematician, Colin Maclaurin (1698—1746) in 1742 and bearing his name gives the expansion of a function of x in the following

manner:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots "$$

Nothing is said here of the fact that this formula is not due to Maclaurin and that it was mentioned by J Stirling before it was mentioned by Maclaurin. What is, however, of greater interest is that less than one-fifth of a page is devoted to the theorems of Taylor and Maclaurin, while more than a page and a half is devoted to the abacus. The great importance of Taylor's formula may be inferred from the following remarks thereon by G. Darboux: "De toutes des découvertes que les Mathématiciens ont faites au cours de XIXe siecle, la plus féconde sans doute est celle que l' on doit á Cauchy relativement á la serie de Taylor et aux condition de sa convergence. On peut dire qu'elle a renouvellé toutes des branches de l' Analyse." *Œuvres de Henri Poincaré*, tome 2, 1916, page 24.

These remarks by Darboux are of great historical interest and imply that a long and interesting article could easily be written on Taylor's theorem and its influence on the development of mathematics during the nineteenth century. The fact that its importance was not recognized by many writers for more than fifty years is also an element of interest, and the relative spaces devoted to it and to the abacus in the Britannica is perhaps not more striking than the fact that less than one page is devoted therein to such a broad subject as the history of mathematics. It must, however, be admitted that if all the mathematical subjects had been treated as fully as one might desire in such a work, the mathematical articles alone would have required more space than was available for the whole work. It is clear that the reader who will reflect on the amount of space devoted, the various subjects and on the choice of the mathematicians who were selected for biographical sketches, as well as on the reliability of the articles which appear in this latest edition of the Britannica, will thus obtain a large amount of insight of great interest which will not come to those who simply consult this work as a source of information. The present article aims to be of some service to the former of these two types of readers.

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A Brief Analysis of Bhaskara's Bijaganita with Historical and Critical Notes.

1. The following problems and theorems of Bháskara have already been dealt with in the pages of this *journal*:

(i) To solve the two simultaneous indeterminate quadratics—

$$x^2 + y^2 + 1 = \square\ ,\quad x^2 - y^2 + 1 = \square\ ,$$

and

$$x^2 + y^2 - 1 = \square\ ,\quad x^2 - y^2 - 1 = \square\ .$$

"If thou art proficient, calculate and state those two numbers the sum and difference of whose squares are squares, being increased or decreased by one." One of these equations is proposed by J. C. Swaminarayan and a sketchy solution of it is given by R. P. Paranjpye in the *Journal*, for Oct. 1909. and a close translation by me of Bhaskara's solution appears in the issue of Dec. 1909. A complete solution of the problem with that of an extension of it and remarks by myself and T. P. Trivedi appear in the Journal for Aug. 1928.

(ii) Remarks on Brahmagupta's formula for the area of the quadrilateral and deduction of that of the triangle with Bhaskara's strictures thereon, appear at the end of the solution of Q. 196, in the Journal for Aug. 1910.

(iii) A note on "Bhaskara and Samçlishta Kuttaka" by A.A.Krishnaswami Ayyangar appears in the Journal for Feb 1929

(iv) Q. 1501 proposed by me in the issue of Feb. 1928, has been completely solved in the Journal for Feb. 1929. It contains the two indeterminate quadratics $x^2 - 67y^2 = 1$, $x^2 - 61y^2 = 1$, of Bhaskara, the close translation of whose solution by H. T. Colebrooke has also been given. Further, I have added notes on Bhaskara's methods, especially the interlinking process called *Bhávaná*, with its two divisions the *samása* (composition) and the *antara* (difference).

2. Besides the complete theory and general solution of the standard indeterminate quadratic, which Bhaskara always writes in the form $ay^2 \pm b = x^2$, he has in the Bija-ganita given many elegant and notable

theorems, with illustrative examples, about general indeterminate equations. I shall not here touch upon equations of the first degree the theory of which is for the most part reproduced in the *Lilavati*; there is also no need to describe his notation which he has given in an accurate and sufficiently intelligible manner.

The most general form of the indeterminate quadratic is $y^2 = ax^2 + bx + c$, and of this almost all varieties have been solved by Bhaskara and general rules are enunciated in verse-stanzas. Taking an illustrative example given by him, $y^2 = 6x^2 + 2x$, we have

$$6y^2 + 1 = 36x^2 + 12x + 1 = (6x + 1)^2.$$

$$\therefore \quad z^2 - 6y^2 = 1,$$

if $z = 6x + 1$. Thus the equation is completely solved from the continued fraction of $\sqrt{6}$. The standard equation is styled by Bhaskara "the multiplied square," *varga* or *kriti-prakriti*, as one of the quadratic variables is multiplied by a constant (See the third line of the sloka quoted in Q. 1501 proposed by me in the *Journal* for Feb. 1928). He has also solved some simultaneous indeterminate quadratics; besides the equations of Q. 156 (for complete solution *vide* the issue for Aug. 1928), Bhaskara has proposed the more difficult pair—

$$7x^2 + 8y^2 = \square,\ 7x^2 - 8y^2 + 1 = \square.$$

By skilful choice of new variables he reduces the question to the solution of $20t^2 + 1 = u^2$.

Many rules for the solution of the standard equation $ax \pm b = y^2$ are given by Bhaskara and some of them have been attributed to older authors; thus the germ of the theory of Quadratic Residues is also to be found in the writings of ancient Indian algebraists As an example take $y^2 - 4 = 7x$; of course we can put $(y + 2)$ and $(y - 2)$ equal to any two numbers whose product shall be $7x$, but the predecessors of Bhaskara give a more practical general method. We have $y^2 = 7x + 4$; putting the latter $= (7z + 2)^2$, we get at once $x = 7z^2 + 4z$ and of course y is $7z + 2$. Thus an infinity of integral values for x and y is obtained by giving to z any rational value whatsoever. Again taking $y^2 - 30 = 7x$, Bhaskara puts $y^2 - 16 = 7x + 14$, thus showing that he knew 2 to be a quadratic residue of 7. Setting $y = 7z + 4$, we get

$$49z^2 + 56z = 7x + 14 \text{ or } x = 7z^2 + 8z - 2,$$

along with $y = 7z + 4$. Other similar and more difficult examples are solved, and in one place, giving in 3 *áryás* the rule for the choice of a new variable, Bhaskara remarks curtly that it was enunciated by previous masters for the instruction of the dull—

(अन्यवर्णं वर्गादिः कल्पनीयः। तत्कल्पनायां मन्दावबोधोर्थम् पूर्वाचार्यैः पठितः)

I shall give only one more example to show the grasp of theory as well as freshness of method of this master. Let us take the simultaneous equations $3x+1=\square$, $5x+1=\square$. It is not necessary to go over the whole process; Bhaskara gets the lowest two solutions; $x = 16$ and $x = 1008$, and indicates an unlimited number of solutions in larger numbers. By means of helpful substitutions he arrives at the equation $u^2 - 15t^2 = 10$. One evident solution of this is $u = 5$, $t = 1$, while one solution of the auxiliary equation $u_1^2 - 15t_1^2 = 1$ is $u_1 = 4$, $t_1 = 1$. By combining these he gets $u = 5.4 + 1.1.15 = 35$, $t = 5.1 + 4.1 = 9$ as the first solution. Again interlinking this solution with $u_1 = 4$, $t_1 = 1$, he further gets $u = 275$, $t = 71$, and says that the process may be repeated any number of times. Of course these solutions can be obtained by combining $u = 5$, $t = 1$ with the values of u_1 and t_1 obtained from the convergents of $15^{\frac{1}{2}}$. The reader's attention is sure to be arrested by the modernness of this method; *Cf.* Hall and Knight: *Higher Algebra* (Edn. IV), §§ 375-376. The interlinking process is technically called *Bhávaná*; it is said to be *Samása* (composition) *Bhávaná* when the plus sign and *antara* (difference) *Bhávaná* when the minus sign is used between the two derived terms. See the Solution of Q. 1501, in the *Journal* for Feb. 1929.

3. It will be seen from the two previous articles that the chief properties of simple continued fractions, non-recurring as well as recurring, given below, with their application to important problems, were all known in India before 1150 A.D., the year in which Bhaskara completed his Siddhánta-Şiromaņi:—

Any quantity, commensurable or surd of the second order, can be expressed as a simple C.F. terminating in the case of a commensurable quantity, non-terminating in that of a quadratic surd. A non-terminating simple C. F. must be incommensurable. The numerators and denominators of the successive convergents obey the law

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^n$$

from which it follows at once that every convergent is in its lowest terms. The odd convergents form an increasing series of rational fractions continually approaching to the value of the whole C. F. ; the even convergents form a decreasing series having the same property. Every even convergent is greater than every odd convergent; every odd convergent is less than, and every even convergent greater than, any following convergent. The solution in integers of the indeterminate equation $ax + by = c$, where a, b, c are rational numbers, may be effected by means of non-recurring continued fractions. The value of a simple recurring C. F. is the positive root of a quadratic equation of which the other root is negative. As the fraction is infinite, its value cannot be commensurable and must be a quadratic surd; conversely, every positive quadratic surd can be expressed as a simple recurring C.F. There is only one non-recurring partial quotient; in the cycle of recurring quotients the last quotient is double of this non-recurring quotient. The recurring quotients are symmetrically arranged about the middle quotient or middle pair of quotients. The solution in integers of the quadratic indeterminate equations $x^2 - Ny^2 = \pm 1$ may be effected by reducing $N^{\frac{1}{2}}$ to a recurring C. F. and taking for x and y the values of the numerator and denominator of the last partial convergent in any cycle when the number of such quotients is odd. If the number of quotients in the cycle is even, only solutions of $x^2 - Ny^2 = +1$ can be found. The solution in integers of $x^2 - Ny^2 = \pm b$ can be effected only if $(-1)^n b$, called by Bhaskara, the *additive*, occurs among the rational denominators in the process of conversion.

Bhaskara's immediate predecessors originated the theory of the recurring C. F. which he completed in the finished form in which it was given by the mathematicians of Western Europe only in the middle of the 18th century, that is to say, six hundred years after Bhaskara. But the theory of the non-recurring C. F. was known to Brahmagupta and was probably originated by his immediate predecessors. The date of completion of Brahmagupta's Siddhánta, generally styled the Brahma-Sphutasiddhánta*, is known to be 638 A.D , so that simple non-recurring

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* This is due to the fact that the word ***sphutam*** (lucidly) occurs in the very first Áryá of this Siddhánta :—

Brahmoktam grahaganitam, mahatá kálena yat khilibhutam |
Abhidhiyate ***sphutam*** tat. Jishnusuta—Brahmaguptena ||.

C.F.'s were handled here as early as the latter half of the sixth century.

(a) It is a matter of great regret that even a mathematician of such high repute as A. E. Jolliffe is not aware of the facts mentioned above, as may be seen from the following extracts from his article on *Continued Fractions* in the Encyclopædia Britannica, XI Edn (1911):—

"In the case of a recurring continued fraction which represents $\sqrt{N}$, where N is an integer, if n is the number of partial quotients in the recurring cycle, and p_{nr}/q_{nr} the nrth convergent, then

$$p^2_{nr} - Nq^2_{nr} = (-1)^{nr},$$

whence, if n is odd, integral solutions of the indeterminate equation $x^2 - Ny^2 = \pm 1$ (the so-called *Pellian equation*) can be found. If n is even, solutions of the equation $x^2 - Ny^2 = +1$ can be found." ... ... "The theory and development of the simple recurring continued fraction is due to Lagrange." ... ... "The invention of continued fractions is ascribed generally to Pietro Antonia Cataldi, an Italian mathematician, who died in 1626. He used them to represent square roots, but only for particular numerical examples, and appears to have had no theory on the subject. A previous writer, Rafaello Bombelli, had used them in his treatise on Algebra (about 1579), and it is quite possible that Cataldi may have got his ideas from him ... . They next appear to have been used by Daniel Schwenter (1585—1636) in a *Geometrica Practica*, published in 1618. He uses them for approximations. The theory, however, starts with the publication in 1655 by Lord Brouncker of the continued fraction $\frac{1}{1+}\frac{1^2}{2+}\frac{3^2}{2+}\frac{5^2}{2+}\cdots$ as an equivalent of $\pi/4$. This he is supposed to have deduced, no one knows how, from Wallis' formula for $4/\pi$, *viz.*, $\frac{3.3.5.5.7.7....}{2.4.4.6.6.8....}$. John Wallis discussing this fraction in his *Arithmetica Infinitorum* (1656), gives many of the elementary properties of the convergents to the general continued

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The Brahma mentioned in the first line was an older astronomer who probably flourished in the latter half of the fourth century and whose astronomical treatise is generally called Brahma-Siddhánta and sometimes Pitamaha-siddhánta on account of its antiquity.

fraction, including the rule for their formation." ... "Nicol Saunderson (1682—1739), Euler and Lambert helped in developing the theory, and much was done by Lagrange in his additions to the French edition of Euler's Algebra (1795)."

(*b*) It is quite evident from various sections of the Bija-Ganita that Bhaskara knew the general rules for the solution of simultaneous indeterminate equations of the first degree. He has treated most of the cases of the indeterminate quadratic equation $y^2 = ax^2 + bx + c$, including the form $y^2 = a^2x^2 + c^2$, whence he deduces rules for finding the sides and hypotenuse of rational right-angled Δs (which were known also to his predecessors) and the form $y^2 = ax^2 \pm b$ (the standard indeterminate quadratic); and has given many practical examples solved by methods of general application.

Arthur Cayley in his article on *Equation* (Ency. Brit. XI) has given the general solution of the indeterminate problem of the second degree for which he takes the equation $y^2 = a + bx + cx^2$ for finding x so that y may be a rational quantity. Most of the cases treated by him are, as will now be perceived, actually given in the Bija-Ganita. The method he gives for solving simple simultaneous indeterminate equations is also very like that adopted by Bhaskara who ascribes it to his predecessors. It is a pity, therefore, that this great mathematician was not aware of the achievements of the ancient Indians in this line of research. Cayley is aware of the fact that Bhaskara recognised the negative root of a quadratic equation and effected its interpretation. But in giving the theory of the indeterminate equation $ax \pm by = c$, he refers to the article on Continued Fractions by Jolliffe which as we have seen makes no mention whatsoever of any Sanscrit mathematician.

(*c*) The method of finding the greatest common factor of two integers and that for solving the congruence $a'x \equiv b'$ (mod. m), given by Prof. G. B. Mathews in his article on *Number* (Ency. Brit. XI), are practically the same as those given by Brahmagupta. Mathews remarks that "An important problem is to find a number which has given residues with respect to a given set of moduli." Examples of such problems with practical and generally applicable methods of solution are to be found in Bhaskara's Bija-Ganita With suitable change of variables the problem is usually reduced to the solution of $ax \pm by = c$, rules for resolving which are given in the *Kuttaka* of the first degree. For illustra-

tion I cite two examples: (1) One person has 5, 8, 7 rubies, sapphires, pearls, respectively, another 7, 9, 6 respectively and the two are equally rich; the total number of the three sorts of gems is 90 with the first and 62 with the second; to find the relative prices of a ruby, a sapphire and a pearl. Bhaskara has given all thirteen solutions. (2) Find two numbers which being divided by 5 and 6 respectively have remainders 1 and 2; also the difference of which divided by 3 has 2 as a residue, the sum of which divided by 9 has 5 as a residue, and their product divided by 7 leaves 6 as remainder. Bhaskara expressly excludes 6 and 8 from the solution, and gives the results

$$N_1 = 315\,h + 51, \quad N_2 = 378\,h + 332,$$

which satisfy all the conditions.

In discussing the reduction of $(a, b, c)\ (x, y)^2$ into itself by means of autómorphic substitutions, Prof. Mathews refers to the integral solution of $t^2 - Du^2 = 4$. Proceeding further he states—"In the case when $D \equiv 0 \pmod{4}$ we have $t = 2T$, $u = 2U$, $D = 4N$, and (T, U) any solution of $T^2 - NU^2 = 1$. This is usually called the *Pellian equation*, though it should properly be associated with Fermat, who first perceived its importance. The minimum solution can be found by converting $\sqrt{N}$ into a periodic continued fraction." The reader will perceive that here too no mention is made of ancient Indian mathematicians for their accurate methods of converting quadratic surds into periodic C F 's and their complete solution of the indeterminate quadratic of the standard form $ay^2 \pm b = x^2$.

4. Although Dr. F. Cajori in his *History of Mathematics* (1906) has done scant justice to Bhaskara himself, his appreciation of the great achievements of ancient Hindu astronomers and mathematicians is both just and generous as will be seen from the following quotations.—

'The following century [the 8th] produced only two names of importance; namely, Cridhara, who wrote a *Ganita-sara* ("Quintessence of Calculation") and Padmanabha, the author of an algebra. The science seems to have made but little progress at this time; for a work entitled *Siddhantaciromani* ("Diadem of an Astronomical System"), written by Bhaskara Açarya in 1150, stands little higher than that of Brahmagupta, written over 500 years earlier. The two most important mathematical

chapters in this work are the *Lilavati* (=" the beautiful ", *i.e.* the noble* science) and *viga-ganita* (=" *root-extraction "), devoted to arithmetic and algebra. From now on, the Hindoos in the Brahmin schools seemed to content themselves with studying the masterpieces of their predecessors. ... In the Hindoo solutions of indeterminate equations, Cantor thinks he can see traces of Diophantine methods. Some technical terms betray their Greek origin. Even if it be true that the Indians borrowed from the Greeks, they deserve great credit for improving and generalising the solutions of linear and quadratic equations. ... We have seen that this very subject [indeterminate analysis] was a favourite with Diophantus, and that his ingenuity was almost inexhaustible in devising solutions for particular cases. But the glory of having invented *general* methods in this most subtle branch of mathematics belongs to the Indians.... Euler's process of reducing a/b to a continued fraction amounts to the same as the Hindoo process of finding the greatest common divisor of a and b by division. This is frequently called the Diophantine method. Hankel protests against this name, on the ground that Diophantus not only never knew the method, but did not even aim at solutions purely integral. ... In the solution of $xy = ax + by + c$, they [the Hindoos] applied the method re-invented later by Euler, of decomposing $(ab + c)$ into the product of two integers m, n and of placing $x = m + b$ and $y = n + a$. Remarkable is the Hindoo solution of the quadratic equation $cy^2 = ax^2 + b$. With great keenness of intellect they recognised in the special case $y^2 = ax^2 + 1$ a fundamental problem in indeterminate quadratics. They solved it by the *cyclic method* ... Doubtless this "Cyclic method" constitutes the greatest invention in the theory of numbers before the time of Lagrange. The perversity of fate has willed it, that the equation $y^2 = ax^2 + 1$ should now be called *Pell's Problem*, while in recognition of Brahmin scholarship it ought to be called the "Hindoo problem" It is a problem that has exercised the highest faculties of some of our greatest modern analysts.'

(*b*) From the remarks which W. W. Rouse Ball makes on the

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* Of course these equivalents are quite faulty. *Lilavati*, the arithmetic, was so named after Bhaskara's daughter; the algebra is named *Bija-ganita* (*viga-ganita*) from the fact that it is the "calculation of the seed, origin or root" *i.e.* of the unknown (avyakta) from known (vyakta) data. The *roots* are not those of arithmetical numbers, but those of the general quadratic and of particular forms of the cubic and the biquadratic.

Bija-ganita as late as 1915* his case would appear to be simply hopeless. He begins his examination of the work by mentioning the commonplace of the threefold signification of the invocation (mangalácharaṇa) † which he terms "a sentence" ingeniously framed. He then describes the algebraical notation framed by Bhaskara and proceeds to give the enunciation and proof of elementary theorems which, as we have seen, are almost all given in the Lilavati. Rouse Ball then dismisses the whole of the latter and larger part of the work with the astounding remark—"He [Bhaskara] then gives the rules of cipher as in the *Lilavati*; solves a few equations; and lastly concludes with some operations on surds. Many of the problems are given in a poetical setting with allusions to fair damsels and gallant warriors." How inapt these observations are will be quite evident from the fact that the "few equations" and "some operations" on surds involve the whole theory of terminating and recurring continued fractions and their application to the solution of the standard indeterminate quadratic; the reduction of most of the cases of the more general indeterminate quadratic; the solution of multiliteral equations and of special forms of cubic and biquadratic equations with numerical co-efficients:—accompanied by difficult examples which in some cases actually adumbrate the theory of quadratic residues.

(*c*) In a trenchant note—' Did the Greeks solve the Quadratic Equation?" (this *Journal*, Aug. 1924), G. A Millar remarks that "the ancient Greeks found formulas by means of which the quadratic equation can be solved, but they did not advance far enough in their algebraic

* "A Short Account of the History of Mathematics," by W. W Rouse Ball, etc. etc. (Sixth Edition ; Macmillan & Co, Ltd., London 1915) In an earlier edition, that of 1888, Rouse Ball had gone to the length of styling the book he was commenting on as "Bjita Ganita"!

† Utpádakam yat pravadanti buddher—
Adhisthitam satpurushena sánkhyáh |
Vyaktasya Kritsnasya tadeka bijam—
Avyaktam – Isam Ganitam cha vande || .

A close and accurate translation of this sloka will be fonnd in Colebrooke's edition of the Siddhanta siromani :—"Algebra with Arithmetic and Mensuration) from the Sanscrit of Brahmegupta and Bháscara, by Henry Thomas Colebrooke, F.R.S., M. Linn and Geol. Soc, etc" London, John Murray. 1817. The Astrological part of the Siddhanta has not been included in the translation.

work to solve the general quadratic equation in one unknown.". The writer emphasises the point that the quadratic was not solved till the real meaning of imaginary numbers was expounded by Argand, Gauss and Hamilton; and states that the modern theory of irrational numbers and the Galois theory of equations are required to pour fresh light on the solution. "We would add that the Greeks, the Indians and the Arabs, taken together, did not solve this equation, although each of them made important contributions towards the solution." These facts may be cheerfully accepted. I have, however, to draw attention to the undoubted conclusion that the early medieval Indians recognised the negative and surd roots of quadratics and that they invented the method of "completing the square," which are almost as great advances on Nicomachus* and Diophantus as the methods of Gauss and Dedekind are on those of Padmanabha and Bhaskara. Everywhere in the discussion of the indeterminate quadratic, $ay^2 + b = x^2$, Bhaskara accepts negative solutions and states that the square root can be both positive and negative. Negative solutions are rejected only in the case of practical problems where such solutions have no standing.

5. Diophantus of Alexandria undoubtedly deserves the title of "father of Algebra". The most probable date† now assigned to him is the middle of the 3rd century, so that his work is anterior to that of Aryabhata by 200 years and to that of Brahmagupta by nearly 400. He has solved problems leading to simple determinate equations in 1, 2, 3 and 4 variables and to determinate quadratic equations. He also solves indeterminate equations of the first degree in one or more variables, which are generally transformed into determinate equations by arbitrarily assuming a value for one of the required quantities. Diophantus is satisfied with a rational, even if fractional, solution and does not always obtain a solution in integers. The problems leading to his indeterminate quadratic equations involve the process of obtaining in rational square numbers one or two linear or quadratic functions of one variable by finding suitable values for that variable. As he has only one symbol for the unknown quantity, his problems often give rise to equations whose roots involve surds or imaginaries. In such cases Diophantus formulates the auxiliary problem for correcting the assumptions of the original problem, and from

* The works of this writer, a Jew by birth, translated with additions by Boetius (Boethius, c. 476–524), formed the staple of European arithmetic and algebra up to the close of the 13th century.

† For many of the statements here made I am indebted to the scholarly work of Sir Thomas L. Health—"Diophantus of Alexandria: A study in the History of Greek Algebra." (Second edition, 1910; Cambridge.)

it obtains an equation, in place of the original one, which can be solved rationally. It is thus evident that his processes lack the polish and generality of those of Brahmagupta and other ancient Indian mathematicians, as was indeed inevitable on account of his antiquity and his defective notation. On the other hand it is his outstanding merit that in the "Porisms" Diophantus has originated the theory of certain elegant indeterminate problems involving the squares and products of several numbers, which awaited generalisation until the times of Fermat and complete investigation until those of Euler and Lagrange.

6. The following extracts from the learned *Dissertation*, prefixed by Colebrooke to his edition of parts of Bhaskara's and Brahmagupta's works, require, I am sure, no excuse for presentation to the careful attention of the reader. Wide culture, high scientific attainments, patient research and sound judgment all go to make Colebrooke's statements exact and accurate in fact and sober and weighty in criticism. It is a matter for both wonder and regret that more than a century ago he had come to unimpugnable conclusions which great mathematicians, writing recently on the theory and history of special branches of our science, have failed to draw.

"The Arabs themselves scarcely pretend to the discovery of Algebra. They were not in general inventors but scholars, during the short period of their successful culture of the sciences; and the germ at least of the Algebraic analysis is to be found among the Greeks in an age not precisely determined, but more than probably anterior to the earliest dawn of civilization among the Arabs: and this science in a more advanced state subsisted among the Hindus prior to the earliest disclosure of it by the Arabians to modern Europe. ... In the actual advanced condition of the analytic art, it is not hoped that this version of ancient *Sanscrit* treatises on Algebra, Arithmetic, and Mensuration, will add to the resources of the art, and throw new light on mathematical science, in any other respect, than as concerns its history. Yet the remark may not seem inapposite, that had an earlier version of these treatises been completed, had they been translated and given to the public, when the notice of mathematicians was first drawn to the attainments of the Hindus in astronomy and in sciences connected with it, some addition would have been then made to the means and resources of Algebra for the general solution of problems by methods which have been re-invented, or have been perfected, in the last age ... There will be in the sequal occasion to show that modes of analysis, and, in particular, general methods for the solution of indeterminate problems both of the first and

second degrees, are taught in *Vija-gaṇita*, and those for the first degree repeated in the *Līlāvati*, which were unknown to the mathematicians of the West until invented anew in the last two centuries by algebraists of France and England. It will be also shown that *Bhaskara*, who himself flourished more than six hundred and fifty years ago, was in this respect a compiler, and took those methods from Indian authors as much more ancient than himself ...Confining the comparison of Hindu and modern Algebras to conspicuous instances, the next for notice is that of the solution of indeterminate problems of the 2nd degree: for which a general method is given by Bhascara, besides rules for subordinate cases: and two general methods (one of them the same with Brahmagupta's) besides special cases subservient however to the universal solution of problems of this nature; and, to obtain whole numbers in all circumstances, a combination of the method for problems of the first degree with that for those of the second, employing them alternately, or, as the Hindu algebraist terms it, proceeding in a circle [by the cyclic method]. ...Bhascara's second method for a solution of the problem on which all indeterminate ones of this [second] degree depend, is exactly the same, which Lord Brouncker devised to answer a question proposed by way of challenge by Fermat in 1657. The thing required was a general rule for finding the innumerable square numbers, which multiplied by a proposed (non-quadrate) number, and them assuming an unit, will make a square. ... But neither Brouncker, nor Wallis, who himself contrived another method, nor Fermat, by whom the question was proposed, but whose mode of solution was never made known by him (probably because he had not found anything better than Wallis and Brouncker discovered), nor Frenicle, who treated the subject without, however, adding to what had been done by Wallis and Brouncker, appear to have been aware of the importance of the problem and its universal use: a discovery, which, among the moderns, was reserved for Euler in the middle of the last century. To him, among the moderns, we owe the remark, which the Hindus had made more than a thousand years before, that the problem was requisite to find all the possible solutions of equations of this sort. Lagrange takes credit for having further advanced the progress of this branch of the indeterminate analysis, so lately as 1767; and his complete solution of equations of the 2nd degree appeared no earlier than 1769....Priority seems then decisive in favour of both Greeks and Hindus against any pretensions on the part of the Arabians, who in fact, however, prefer none, as inventors of Algebra. They were avowed borrowers in science: and, by their own unvaried acknowledgment, from the Hindus they learnt the science of numbers. That they also received

the Hindu Algebra, is much more probable, than that the same mathematician who studied the Indian arithmetic and taught it to his Arabian brethren, should have hit upon Algebra unaided by any hint or suggestion of the Indian analysis. The Arabs became acquainted with the Indian astronomy and numerical science, before they had any knowledge of the writings of the Grecian astronomers and mathematicians....But the age of the earliest known Hindu writer on Algebra, not being with certainty carried to a period anterior, or even quite equal, to that in which Diophantus is on probable grounds placed, the argument of priority, so far as investigation has yet proceeded, is in favour of Grecian invention. The Hindus, however, had certainly made distinguished progress in the science, so early as the century immediately following that in which the Grecian taught the rudiments of it. The Hindus had the benefit of a good arithmetical notation; the Greeks, the disadvantage of a bad one. Nearly allied as algebra is to arithmetic, the invention of the algebraic calculus was more easy and natural where arithmetic was best handled. No such marked identity of the Hindu and Diophantine systems is observed, as to demonstrate communication. They are sufficiently distinct to justify the presumption, that both might be invented independently of each other. If, however, it be insisted that a hint or suggestion, the seed of their knowledge, may have reached the Hindu mathematicians immediately from the Greeks of Alexandria, or mediately through those of Bactria, it must at the same time be confessed, that a slender germ grew and fructified rapidly, and soon attained an approved state of maturity in Indian soil."

7. The summary which has been furnished above of the rules and processes expounded in the *Bija-ganita*, the opinions thereon which have been cited and the probable dates assigned, after careful examination, to ancient mathematicians, will make it patent to the thoughtful reader that, if the science of Algebra was originated by the ancient Greeks, it was soon after brought to a high pitch of development by the ancient Indians. In conclusion, therefore, I beg to draw the earnest attention of the Committee of our Society to the great importance and urgent necessity of bringing these facts to the notice of mathematicians in India as well as in Western Europe and the U.S. of America. Above all, as Bhaskara has made the equation $ay^2 \pm 1 = x^2$ all his own, it would be only just and proper if by common consent the standard indeterminate quadratic equation is designated *Bhaskara's Equation* in honour of this great Indian master (Acharya).

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Solutions.

Question 1547.

A Prize Problem.

(V. Ramaswami Aiyar, M.A.):—Show that the number of ways of connecting n villages with one another by $n-1$ roads, each of them connecting two of the villages directly, is n^{n-2}.

1.

Solution by A. A. Krishnaswami Aiyangar, M.A., L.T., Intermediate College, Mysore.

1. Replacing, for the sake of convenience, the n villages by n points $A_1, A_2, \dots A_n$, the problem may be stated as follows:—

In how many different ways can the set of n points $A_1, A_2, \dots A_n$ be connected by **a net of n—1 straight lines,** *so that every pair of points, say, A_r, A_s, may be connected, either directly by the straight line A_rA_s or by a series of straight lines, $A_rA_l \dots A_kA_s$?*

Properties of the net.

(i) A fundamental fact to be noted is that in any mode of connecting the n points, there is *one* route between any two points; there is also *only one* such route; in other words there is no *closed circuit* forming part of the net.

(ii) From any point A_r, there may be more than one straight line emanating, but of these *only one*, say A_rA_s, will form part of the path leading from A_r to A_n and we shall consider this as *the straight line associated with* A_r, A_s lying between A_r and A_n in this path. In this way, we may associate with each of the $(n-1)$ points $A_1, A_2, \dots A_{n-1}$ (A_n alone excepted) a distinct straight line. It is clear that none of these lines can be associated with more than one point; for, if A_rA_s be associated with both A_r and A_s, it means that A_s lies between A_r and A_n and at the same time A_r is also between A_s aad A_n which is impossible unless we comtemplate a closed circuit of which A_r, A_s, A_n are vertices, which is excluded by (i).

(iii) Further, any straight line, say $A_p\ A_q$ of the net must be associated with one or other of its extremities; for, there must exist, according

to our hypothesis, a unique route from A_p to A_n and if A_q does not lie in this route, then $A_q A_p \ldots A_n$ must be the unique route from A_q to A_n and $A_p A_q$ is therefore associated with A_q.

(iv) From (ii) and (iii), we infer that there are exactly $(n-1)$ straight lines in our net and these may be regarded as associated with the vertices $A_1, A_2 \ldots A_{n-1}$ respectively.

2. Symbolic Representation.

The scheme of straight lines in any net may be represented symbolically by

$$\begin{Bmatrix} A_1 & A_2 & \ldots & A_{n-1} \\ A_r & A_s & \ldots & A_t \end{Bmatrix}$$

or

$$\begin{Bmatrix} 1, & 2, & \ldots & \overline{n-1} \\ r, & s, & \ldots & t \end{Bmatrix} \quad (\alpha)$$

dropping the A's; which means that $A_1 A_r$, $A_2 A_s$, ... $A_{n-1} A_t$ are respectively associated with $A_1, A_2, \ldots A_{n-1}$. Each symbol as in (α) can represent only one net and gives immediately its geometrical structure. Conversely, the geometrical structure of a net enables one readily to write down its appropriate symbol.

Obviously, the elements in the same column of (α) cannot be identical; thus $r \neq 1$, $s \neq 2$, and so on. Further, (α) regarded as a quasi-substitution symbolism cannot contain a cyclic substitution; for a cycle, say

$$\begin{Bmatrix} r & s & \ldots & m \\ s & l & \ldots & r \end{Bmatrix}$$

implies a corresponding closed circuit $A_r A_s A_l \ldots A_m$ which the net certainly does not possess.

Therefore, there must be repeated elements in the second row of (α) and since A_n (or n) does not occur in the first row, it must occur in the lower.

3. Let $A_1 A_r$, $A_1 A_l$, ... be σ_1 straight lines passing through A_1, of which $A_1 A_r$ alone is associated with A_1 and therefore the other lines must be associated with A_l, ... Thus A_1 (or 1) will repeat $\sigma_1 - 1$ times in the second row.

Similarly, if there are $\sigma_2, \sigma_3, \ldots \sigma_{n-1}$ and σ_n lines of the net, passing through $A_2, A_3, \ldots A_{n-1}, A_n$ respectively, A_2 (or 2), A_3 (or 3), ... A_{n-1} (or $n-1$) will repeat $\sigma_2 - 1, \sigma_3 - 1, \ldots \sigma_{n-1} - 1$ times, while A_n (or n) will repeat σ_n times in the second row of (α), which accommodates $(n-1)$ elements in all. Hence

$$(\sigma_1 - 1) + (\sigma_2 - 1) + (\sigma_3 - 1) + \ldots + (\sigma_{n-1} - 1) + \sigma_n = n - 1,$$

i.e.,
$$\sigma_1 + \sigma_2 + \ldots \sigma_n = 2n - 2. \qquad \ldots \text{ I}$$

4. We will now show that for any particular set of values of $\sigma_1, \sigma_2, \ldots \sigma_n$ satisfying I, we can construct

$$\frac{(n-2)!}{(\sigma_1 - 1)!\,(\sigma_2 - 1)! \ldots (\sigma_n - 1)!}$$

different symbols of the type (α) corresponding to an equal number of permutations of the elements $(1, 2, \ldots n)$, occurring respectively $\sigma_1 - 1$, $\sigma_2 - 1, \ldots \sigma_n$ times in the second row of (α), subject to the conditions of para 2.

The $(\sigma_1 - 1)$ elements (A_1) of (α) can occupy any of the $(n-2)$ places under $(A_2, \ldots A_{n-1})$, the place under A_1 alone being forbidden. This can happen in ${}^{n-2}C_{\sigma_1 - 1}$ ways. Having fixed the elements (A_1) duly, we are left apparently with $(n - \sigma_1)$ places for accommodating (A_2); but of these, *one* place is again tabooed, the place under either A_1 or A_2, the former to avoid the cycle $(A_1, A_2 / A_2, A_1)$ if A_1 has already come under A_2 and the latter, if otherwise. Thus, we have only $(n - \sigma_1 - 1)$ places free for the occupation of $(\sigma_2 - 1)$ elements (A_2), which can now be disposed of in ${}^{n-\sigma_1-1}C_{\sigma_2 - 1}$ ways.

Clearly, the sets of elements $A_3, A_4, \ldots A_{n-1}$ can be treated in the same manner. For, suppose all the sets of elements $A_1, A_2, \ldots A_{r-1}$ have been fixed up in their due places and we have next to accommodate the elements (A_r); then we have at our disposal

$$[n - 1 - (\sigma_1 - 1 + \sigma_2 - 1 + \ldots \sigma_{r-1} - 1]$$

empty places, out of which *one* again is taboo for A_r, *viz.*, the place under A_r if A_r is free or that under A_k $(k \leqslant r-1)$ which will complete the cycle

$$\left\{ \begin{matrix} A_r & A_p & \ldots & A_k \\ A_p & A_t & \ldots & A_r \end{matrix} \right\}$$

where $(A_p, A_q, \ldots A_k)$ belong to the set $(A_1, A_2, \ldots A_{r-1})$. After the elements $A_1, A_2, \ldots A_{n-1}$ have been placed, there are left exactly as many places as there are elements (A_n).

Therefore, the total number of ways in which we can form the required permutations for the specification $(\sigma_1, \sigma_2, \ldots \sigma_n)$ *is*

$$\overset{n-2}{\underset{\sigma_1-1}{C}} \cdot \overset{n-\sigma_1-1}{\underset{\sigma_2-1}{C}} \ldots \overset{n-2-(\sigma_1-1+\sigma_2-1+\ldots+\sigma_{n-2}-1)}{\underset{\sigma_{n-1}-1}{C}}$$

i.e., $$\frac{(n-2)!}{(\sigma_1-1)!\,(\sigma_2-1)!\ldots(\sigma_{n-1}-1)!\,(\sigma_n-1)!} \quad \ldots \quad \text{II}$$

5. If we represent a net corresponding to the specification $(\sigma_1, \sigma_2, \ldots \sigma_n)$ by the symbol

$$A_1^{\sigma_1-1} A_2^{\sigma_2-1} \ldots A_n^{\sigma_n-1},^{*}$$

the fact that there are

$$\frac{(n-2)!}{(\sigma_1-1)!\,(\sigma_2-1)!\ldots(\sigma_n-1)!}$$

of them may be indicated by

$$\frac{(n-2)!}{(\sigma_1-1)!\,(\sigma_2-1)!\ldots(\sigma_n-1)!} A_1^{\sigma_1-1} A_2^{\sigma_2-1} \ldots A_n^{\sigma_n-1}.$$

Comparing this with the corresponding term in the multinomial expansion

$$(a_1 + a_2 + \ldots a_n)^{n-2}$$

we at once deduce that the total number of all possible nets is equal to the sum of the co-efficients in this expansion, *viz.* n^{n-2}.

Thus, *the problem is solved.*

6. We now proceed to a *generalization.*

Instead of considering n isolated *points*, we shall consider n isolated *systems* $S_1, S_2, \ldots S_n$, each *system being so called when it contains a set of points connected with one another by a net work of the same specification as in our original problem.* Thus, S_r contains a_r points

* When any of the σ's be equal to 1, say $\sigma_r = 1$, then instead of writing A^0_r, we may well omit it in the symbol $A_1^{\sigma_1-1} A_2^{\sigma_2-1} \ldots,$

connected by $a_r - 1$ straight lines, no two points being connected by more than one route. We may now investigate in how many ways these n systems may be combined into a single system of connected points.

Evidently, in the comprehensive system there must be *only one line* connecting any two of the given systems, say S_k and S_l for two such connecting straight lines will imply a circuit. Thus, the given systems play the same role as isolated points and any scheme of connecting lines of the given systems may be symbolised as in para 2 by

$$\left\{ \begin{matrix} \alpha_1 & \alpha_2 & \dots \dots & \alpha_{n-1} \\ \alpha_r & \alpha_s & \dots \dots & \alpha_t \end{matrix} \right\} \qquad \dots \quad (\beta)$$

where α_k denotes any point of the system S_k; and each symbol (β) designates not one system but $(a_1 . a_r . a_2 . a_s \dots a_{n-1} . a_t)$ systems, since we may connect a point of S_1 with a point of S_r in $a_1 . a_r$ ways and similarly the points of the other pairs of systems in the different columns of (β).

If in the symbol (β), the elements $\alpha_1, \alpha_2, \dots \alpha_n$ occur respectively $\sigma_1, \sigma_2, \dots \sigma_n$ times where $\Sigma \sigma_n = 2n - 2$, each symbol (β) stands for

$$a_1^{\sigma_1} \; a_2^{\sigma_2} \dots \dots a_n^{\sigma_n}$$

different systems and there are

$$\frac{(n-2)!}{(\sigma_1 - 1)! \; (\sigma_2 - 1)! \dots (\sigma_n - 1)!}$$

of symbols (β) for the given σ - specification.

Therefore the total number of ways of combining for all possible σ - specifications is

$$\sum \frac{(n-2)!}{(\sigma_1 - 1)! \; (\sigma_2 - 1)! \; \dots \; (\sigma_n - 1)!} \, a_1^{\sigma_1} . a_2^{\sigma_2} \dots a_n^{\sigma_n}$$

i.e., $$a_1 a_2 \dots a_n (a_1 + a_2 + \dots + a_n)^{n-2} \qquad \dots \quad \text{III}$$

This represents the total number of ways in which $S_1, S_2, \dots S_n$ may be combined into a single system.

Putting $a_1 = a_2 \dots = a_n = 1$, the result of the Prize Problem follows.

2.

Solution by Prof. G. N. Watson, Sc.D, F.R.S., Birmingham.

Let the required number of ways be u_n. The solution will be obtained by constructing a formula which expresses u_{n+1} in terms of $u_1, u_2, u_3, \ldots, u_n$, (evidently $u_2 = 1$, $u_3 = 3$; and u_1 is to be conventionally interpreted as 1 for reasons given later), and then deducing from this formula that, if $u_3, u_4, \ldots, u_n$ are equal to $3^1, 4^2, \ldots, n^{n-2}$, the value of u_{n+1} must be $(n+1)^{n-1}$; the required result is then evident by induction.

First we notice that, when a permissible set of roads has been constructed, no group of them can form a *closed circuit*; for, if r of the roads did so, they would link up r villages only and (when the number of villages is n) the remaining $n - r - 1$ roads would be one too few to link up the remaining $n - r$ villages with each other and with one of the first r villages.

Next let us proceed to enumerate the u_{n+1} ways of linking up $n+1$ villages; suppose that a permissible set of n roads has been constructed and consider those of the roads which have an end at one particular village, A; there will be at least one such road, and there may be more. Let all the villages which can ultimately be reached by starting from A along one particular road be called a *confederation*; for example, if there are ten villages A, B, C, ..., I, J, then, when the roads are AB, BC, BD, AE, EF, FG, AH, HI, AJ, there are four confederations, the first being B, C, D, the second being E, F, G, the third being H, I, and the fourth being J only.

No village can belong to two confederations, for, if it did, it could be reached by two different paths from A, and so a set of roads would form a closed circuit, which is impossible.

It can now be seen that u_{n+1} is the total number of different ways in which the following operations can be effected—(i) grouping the n villages (other than A) into confederations in all possible ways, (ii) in each grouping, joining up the villages of each confederation by roads one less in number than the number of villages in that confederation, and (iii) joining A in all possible ways to one member of every confederation; for each way of effecting these operations gives rise to one permissible set of roads, and conversely.

Suppose now that one particular grouping has been effected, and that in this grouping there are m confederations in all, there being a confederations which each contain r villages, b confederations which each contain s villages, c confederations which each contain t villages, and so on, $r, s, t, \ldots$ being all unequal; then

$$a + b + c + \ldots = m, \qquad \ldots \quad (1)$$

$$ra + sb + tc + \ldots = n; \qquad \ldots \quad (2)$$

since $r, s, t, \ldots$ are greater than or equal to 1, it is evident from these equations that $m \leqslant n$; in fact $m = n$ only when there are n confederations each consisting of one village.

Now the villages in each of the first a confederations can be linked up in u_r permissible ways (and similarly for the other confederations) while A can be joined to a village in any particular one of these a confederations in r ways; these operations being independent, the total number of ways of constructing the roads, when the villages are grouped into the particular set of confederations under consideration, is

$$(ru_r)^a \, (su_s)^b \, (tu_t)^c \ldots \ldots$$

It is at this stage that we see that the conventional meaning to be attached to u_1 is 1, by considering a confederation which consists of a single village.

Next observe that the number of different ways of effecting the grouping so that there are a confederations each containing r villages, b confederations each containing s villages and so on, where $a, b, c, \ldots$, $r, s, t, \ldots$ are a given set of solutions of (1) and (2), is

$$\frac{n!}{(r!)^a \, (s!)^b \, (t!)^c \ldots a! \; b! \; c! \ldots}.$$

This may be seen by enumerating the villages in all the possible $n!$ ways and then taking each particular one of these arrangements and selecting the first r as the first confederation, the next r as the second and so on; the factors in the denominator have then to be inserted because, in the groupings now requisite, attention is not to be paid to the order of the villages in each confederation, nor to the order of the first a confederations, the order of the next b confederations and so on.

Hence the total number of ways of constructing the roads joining the $n+1$ villages (namely u_{n+1}) is the sum of all expressions of the form

$$\frac{n!\,(ru_r)^a\,(su_s)^b\,(tu_t)^c\ldots}{(r!)^a\,(s!)^b\,(t!)^c\ldots a!\;b!\;c!\ldots}$$

where the summation is effected by taking $r, s, t, \ldots$ to be any selection of any number of the integers 1, 2, 3, ... whose sum, by reason of (2), does not exceed n, no integer being repeated in any one selection, and then choosing $a, b, c, \ldots$ so that (2) is satisfied; this is to be done in all possible ways, and (1) may then be regarded as defining m.

Now this sum is the co-efficient of x^n in

$$\sum \frac{n!}{a!\,b!\,c!\ldots}\left(\frac{u_r x^r}{(r-1)!}\right)^a\left(\frac{u_s x^s}{(s-1)!}\right)^b\left(\frac{u_t x^t}{(t-1)!}\right)^c\ldots$$

where the summation is to be effected by taking $r, s, t, \ldots$ to be any selection of any number of the integers 1, 2, 3, ... (no integer being repeated in any one selection), and then choosing $a, b, c, \ldots$ to be any positive integers whose sum does not exceed n; this is to be done in all possible ways, without regard for (2), which is taken into account by means of the selection of the co-efficient of x^n only. Since terms for which $m > n$ would not contribute to the co-efficient of x^n, this sum may be replaced by

$$n!\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{a,\,b,\,c,\,\ldots}\;\sum_{r,\,s,\,t,\,\ldots}\frac{m!}{a!\;b!\;c!\ldots}$$

$$\left(\frac{u_r x^r}{(r-1)!}\right)^a\left(\frac{u_s x^s}{(s-1)!}\right)^b\left(\frac{u_t x^t}{(t-1)!}\right)^c\ldots,$$

where it may be supposed that $r, s, t, \ldots$ are chosen as before and $a, b, c, \ldots$ are now to range over all positive integral values (zeros being now included) whose sum is m.

By the multinomial theorem this is equal to

$$n!\sum_{m=0}^{\infty}\frac{1}{m!}\left[\sum_{p=1}^{n}\frac{u_p x^p}{(p-1)!}\right]^m$$

Consequently u_{n+1} is equal to the co-efficient of x^n in the expansion of

$$n!\ \exp.\left[\sum_{p=1}^{n}\frac{u_p x^p}{(p-1)!}\right];$$

and this is the formula which expresses u_{n+1} in terms of $u_1, u_2, u_3, \ldots u_n$.

To establish the induction we now observe that, if

$$u_1 = u_2 = 1,\ u_3 = 3^1,\ u_4 = 4^2, \ldots,\ u_n = n^{n-2},$$

then we have

$$u_{n+1} = \text{co-efficient of } x^n \text{ in } n!\ \exp.\left[\sum_{p=1}^{n}\frac{p^{p-2}x^p}{(p-1)!}\right]$$

$$= \text{co-efficient of } x^n \text{ in } n!\ \exp.\left[\sum_{p=1}^{\infty}\frac{p^{p-2}x^p}{(p-1)!}\right],$$

the terms for which $p > n$ not affecting the co-efficient of x^n; it has now to be supposed that $|x| < e^{-1}$, to secure convergence.

Next, by an expansion due to Legendre, *Exercises de Calcul Intégral*, II (1816), p. 234, if y is that root of the equation

$$ye^{-y} = x$$

which has the smallest modulus, then

$$e^y = 1 + \frac{x}{1!} + \frac{3^1x^2}{2!} + \frac{4^2x^3}{3!} + \frac{5^3x^4}{4!} + \ldots = \sum_{p=1}^{\infty}\frac{p^{p-2}x^{p-1}}{(p-1)!},$$

so that

$$n!\ \exp.\left[\sum_{p=1}^{\infty}\frac{p^{p-2}x^p}{(p-1)!}\right] = n!\ \exp.(xe^y) = n!\ \exp.(y)$$

$$= n!\sum_{p=1}^{\infty}\frac{p^{p-2}x^{p-1}}{(p-1)!}.$$

The co-efficient of x^n in the last expression is obtained by taking $p = n + 1$, and consequently, it is $(n+1)^{n-1}$. This establishes the induction which has been seen to be adequate to prove the theorem.

3.

Solution by Hans R. Gupta, M.A., Government Intermediate College, Hoshiarpur (Punjab).

1. Mr. Aiyar's Problem is a particular case of the following theorem : —

If there be a set M of m points which are connected with one another in the manner indicated in the problem and a second set N of n unconnected points, the points of set M can be connected with those of set N by n lines in $m(m+n)^{n-1}$ ways.

2. We proceed to give an inductive proof of the general theorem.

Let (m, n) denote the number of ways.

Suppose the theorem is true when the number of points in set M is $\leqslant m$, and that in set N $\leqslant n$, so that,

$$(p, q) = p(p+q)^{q-1};\ p \leqslant m,\ q \leqslant n.$$

(i) To the points of set M, add a new point.

This new point can be connected with r of the n points of set N in ${}^nC_r\,(1, r)$ ways. The m points can now be connected with the remaining $(n-r)$ points of N in $(m, n-r)$ ways.

$$\text{Hence } (1+m, n) = \sum_{r=0}^{r=n} {}^nC_r(1, r)\,(m, n-r);$$

where $(1, 0) = (2, 0) = (3, 0) = \dots = (m, 0) = 1.$

$$\therefore\ (1+m, n) = \sum_{r=0}^{r=n} {}^nC_r\,(1+r)^{r-1}\,.\,m\,.\,(m+n-r)^{n-r-1}$$

$$= \sum_{r=0}^{r=n} {}^nC_r\,.\,(1+r)^{r-1}\,\{(m+n-r)^{n-r} - (n-r)(m+n-r)^{n-r-1}\}$$

[*Lemma* 1. To sum the series on the right-hand side, suppose

$$(x+1)^p - p(x+1)^{p-1} \equiv \sum_{r=0}^{p} {}^pC_r\,.\,(r+1)^{r-1}\,\{(x-r)^{p-r} - (p-r)(x-r)^{p-r-1}\}$$

for positive values of p up to and inclusive of p (an integer).

Let $R \equiv (x+1)^{p+1} - (p+1)(x+1)^p$

$$- \sum_{r=0}^{p+1} {}^{p+1}C_r . (r+1)^{r-1} \{ (x-r)^{p-r+1} - (p-r+1)(x-r)^{p-r} \};$$

then $$\frac{dR}{dx} = (p+1) \{ (x+1)^p - p(x+1)^{p-1}$$

$$- \sum_{r=0}^{p} {}^{p}C_r . (r+1)^{r-1} [(x-r)^{p-r} - (p-r)(x-r)^{p-r-1}] \}$$

$= 0$, by supposition.

$\therefore$ R is independent of x.

Put $x = p+1$, in R,

then $$R = (p+2)^{p+1} - (p+1)(p+2)^p$$

$$- \{ (x-p-1) \sum_{r=0}^{p} {}^{p+1}C_r . (r+1)^{r-1} (x-r)^{p-r} + {}^{p+1}C_{p+1} (p+2)^p \}_{x=p+1} = 0.$$

$$\therefore \quad (x+1)^{p+1} - (p+1)(x+1)^p$$

$$= \sum_{r=0}^{p+1} {}^{p+1}C_r . (r+1)^{r-1} . \{ (x-r)^{p-r+1} - (p-r+1)(x-r)^{p-r} \}$$

Now, the supposition is at once seen to hold when $p = 1, 2, 3$.

$\therefore$ By induction it must hold for all positive integral values of p.]

In the result of the above lemma, put $m+n$ for x and n for p,

then $$\sum_{r=0}^{n} {}^{n}C_r . (r+1)^{r-1} . \{ (m+n-r)^{n-r} - (n-r)(m+n-r)^{n-r-1} \}$$

$$= (m+n+1)^n - n(m+n+1)^{n-1}$$

$$= (m+1)(m+n+1)^{n-1}.$$

$\therefore \quad (m+1, n) = (m+1)(m+n+1)^{n-1}$.

Now, simple induction establishes the theorem of 1 for all values of m.

(ii) To the points of set N, add a new point

This new point can be connected with r of the n points of set N in ${}^{n}C_r (1, r)$ ways. These $(r+1)$ points can be connected with one of the m points of set M in $m. (r+1, 1)$ ways. The m points can now be connected with the remaining $(n-r)$ points of N in $(m, n-r)$ ways.

Hence $(m, n + 1) = \sum_{r=0}^{n} {}^nC_r\, (1, r) . m . (r + 1, 1) . (m, n - r)$.

$$= \sum_{r=0}^{n} {}^nC_r . (1 + r)^r . m^2 . (m + n - r)^{n-r-1},$$

$$= m\,(m + n + 1)^n, \text{ by Lemma 2.}$$

Thus simple induction establishes the theorem for all values of n. The theorem is thus universally true.

[*Lemma* 2. Suppose

$$(x + 1)^n \equiv (x - n) . \sum_{r=0}^{n} {}^nC_r . (r + 1)^r\, (x - r)^{n-r-1}$$

$$\equiv \sum_{r=0}^{n-1} {}^nC_r . (r + 1)^r \{ (x-r)^{n-r} - (n-r)\, (x-r)^{n-r-1} \} + (n+1)^n,$$

for all positive integral values of $n \leqslant p$.

Let $R = (x + 1)^{p+1}$

$$- [\sum_{r=0}^{p} {}^{p+1}C_r . (r + 1)^r \{ (x-r)^{p-r+1} - (p-r+1)(x-r)^{p-r} \} + (p+2)^{p+1}].$$

Then $\dfrac{dR}{dx}$

$$= (p+1) \{ (x+1)^p - \sum_{r=0}^{p} {}^pC_r . (r+1)^r . \{ (x-r)^{p-r} - (p-r)\, (x-r)^{p-r-1} \} \}$$

$= 0$ by supposition.

$\therefore$ R is independent of x.

Put $x = (p + 1)$ in R.

Then $R = 0$.

Proceeding as in Lemma 1, the supposition is established for all positive integral values of n.

Putting $m + n$ for x, we get

$$(m + n + 1)^n = m . \sum_{r=0}^{n} {}^nC_r . (r + 1)^r\, (m + n - r)^{n-r-1}].$$

3. Mr. Aiyar's result is now obtained by putting 1 for m and $n - 1$ for n in the general theorem proved above.

The required number of ways $= (1, n - 1) = n^{n-2}$.

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LIST OF JOURNALS & BOOKS RECEIVED IN THE LIBRARY

during the months of February and March 1930.

1 Abhandlungen aus dem mathematischen seminar der Hamburgischen Universitat, **7**, 4.
2 Acta Mathematica, **53**, 3, 4.
3 American Journal of Mathematics, **52**, 1.
4 American Mathematical Monthly, **36**, 10. and **37**, 1. (3 copies).
5 Annales de l'Ecole Normale Superieure, **65**, 11, 12.
6 Annales of Mathematics, **31**, 1.
7 Astrophysical Journal, **70**, 3, 4.
8 Bulletin of the American Mathematical Society, **35**, 6. **36**, 1, 2.
9 Bulletin of the Calcutta Mathematical Society, **21**, 3, 4.
10 Bulletin des Sciences Mathematiques, **53**, Nov. **53**, Dec. **54**, Jan.
11 Crelle's Journal, **161**, 3, 4.
12 Jahresbericht der deutschen mathematikar Vereinigung, **38**, 9 to 12.
13 Japanese Journal of Mathematics, **6**, 3.
14 Mathematical Gazette, **15**, 205. (3 copies)
15 Mathematische Annalen, **102**, 4, 5
16 Messenger of Mathematics, **58**, 7, 8. (2 copies).
17 Monthly Notices of the Royal Astronomical Society, **89**, 5 to 9. **90**, 1, 2, 3.
18 Philosophical Magazine, **9**, 55, 56.
19 Philosophical Transactions of the Royal Society, **229**, pp. 49—86,
20 Popular Astronomy, **38**, 1, **37**, 7.
21 Proceedings of the Edinburgh Mathematical Society, **2**, 1.
22 Proceedings of the Cambridge Philosophical Society, **26**, 1.
23 Proceedings of the London Mathematical Society, **30**, 4, 5, 6. (2 copies each)
24 Proceedings of the Physico-Mathematical Society of Japan, **11**, 11. **12**, 1. (2 copies each).
25 Proceedings of the Royal Society of London, **126**, 801. (2 copies), 802. (2 copies each).
26 Tohoku Mathematical Journal, **32**, 1, 2.

Books.

1 English Colonial policy and the North-American fishing industry, 1498—1713, by Charles Burnet.
2 On the Spectra of Boron and Aluminum, by W. P. Lansing.
3 Economic effects and social control of intercorporate relationships in public utilities, by Corliss Doran Anderson.
4 Agriculture in Illinois during the Civil period, 1850—1870, by Russell Howard Anderson.

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